

Inverse Systems of Zero-dimensional Schemes in \mathbb{P}^n

Young Hyun Cho

Department of Mathematics, Seoul National University, Seoul 151-742, Korea.

Anthony Iarrobino

Department of Mathematics, Northeastern University, Boston, MA 02115, USA.

April 10, 2012

Abstract

The authors construct the global Macaulay inverse system L_3 for a zero-dimensional subscheme \mathfrak{Z} of projective n -space \mathbb{P}^n over an algebraically closed field \mathbf{k} , from the local inverse systems of the irreducible components of \mathfrak{Z} . They show that when \mathfrak{Z} is locally Gorenstein a generic element F of degree d apolar to \mathfrak{Z} determines \mathfrak{Z} if d is larger than an invariant $\beta(\mathfrak{Z})$. As a consequence of this globalization, they show that a natural upper bound for the Hilbert function of Gorenstein Artin quotients of the coordinate ring of \mathfrak{Z} is achieved for large socle degree. They also show the uniqueness of generalized additive decompositions of a homogeneous form into powers of linear forms, under suitable hypotheses.

The main tools are elementary, but delicate. They involve a careful study of how to homogenize a local inverse system and of the behavior of the homogenization under a change of coordinates. ¹

1 Introduction.

We study Macaulay's inverse systems for the defining ideals of a zero-dimensional (punctual) scheme \mathfrak{Z} of the projective space \mathbb{P}^n over an algebraically closed field \mathbf{k} . Of course, we may suppose that such schemes are contained in an affine subspace \mathbb{A}^n of \mathbb{P}^n . For any graded ideal I in the coordinate ring $R = \mathbf{k}[x_1, \dots, x_{n+1}]$ of \mathbb{P}^n , Macaulay's inverse system I^{-1} is an R -submodule of the dual ring, the divided power series ring $\Gamma = \mathbf{k}_{DP}[X_1, \dots, X_{n+1}]$ and I^{-1} contains the same information as is in the original ideal. Thus, it is not hard to determine which inverse systems arise from zero-dimensional schemes (Proposition 1.13), or which arise from a zero-dimensional scheme concentrated at a single point (Lemma 2.1).

Our main work here begins with an Artinian quotient $A = R'/J$ of the coordinate ring $R' = \mathbf{k}[y_1, \dots, y_n]$ of affine n -space $\mathbb{A}^n \subset \mathbb{P}^n : x_{n+1} = 1$ that defines a zero-dimensional subscheme $\mathfrak{Z} \subset \mathbb{A}^n$, concentrated at a finite set of points. Its *local*, or affine inverse system $L'(J)$ is an R' -submodule of the completion $\hat{\Gamma}'$ of the divided power ring $\Gamma' = \mathbf{k}_{DP}[Y_1, \dots, Y_n]$ dual to R' — this completion is the R' -injective envelope of \mathbf{k} . We then determine from $L'(J)$ the *global* inverse system $L_3 =$

¹**2010 Mathematics Subject Classification:** Primary: 14N05; Secondary: 13N10, 13D40, 13H10, 14C05.

keywords: Macaulay inverse system, regularity degree, globalization, zero-dimensional scheme, Gorenstein Artin ring, irreducible components, generalized additive decomposition.

$(I_3)^{-1} \subset \Gamma$ over \mathbb{P}^n of the defining ideal $I_3 \subset R$ for \mathfrak{Z} . Our goal is to write *generators* of the global inverse system L_3 , in terms of generators of the local inverse systems of the irreducible components of \mathfrak{Z} .

Let the zero-dimensional scheme $\mathfrak{Z} \subset \mathbb{A}^n$ have degree s . Then the ring $A = R'/J$ has dimension s as \mathbf{k} -vector space. The local, or affine inverse system $L'(J)$ also has dimension $\dim_{\mathbf{k}} L'(J) = s$. Since $J = \cap_k J(k)$, the intersection of its primary components, the inverse system $L'(J)$ is a direct sum of the local inverse systems $L'(J(k)) = L'(J\mathcal{O}_{p(k)}) \subset \hat{\Gamma}'$ at the points $p(k)$ of support of \mathfrak{Z} . The scheme \mathfrak{Z} has a unique saturated global defining ideal $I_3 \subset R$, and the coordinate ring $\mathcal{O}_3 = R/I_3$ has Krull dimension one. The global Hilbert function $H_3 = H(\mathcal{O}_3)$, satisfies

$$H_3 = (1, \dots, s, s, s, \dots),$$

the first difference ΔH_3 — often called the h -vector of \mathfrak{Z} — is an O -sequence of total length s (Theorem 1.12). The global inverse system L_3 is a non-finitely generated, graded R -submodule of Γ , whose Hilbert function $H(L_3)$ satisfies $H(L_3) = H_3$. Suppose now that \mathfrak{Z} is concentrated at a single point p . Since $(H_3)_i = s$ for $s \geq \tau(\mathfrak{Z})$, an invariant of \mathfrak{Z} , it is natural to expect that $(L_3)_i$ should be a homogenization of $(L'(J))_{\leq i}$ where $L'(J) = (L_3)_{x_{n+1}=1}$, at least for $i \geq \alpha(\mathfrak{Z})$, the *socle degree* of A . This is what occurs (Proposition 2.11, Theorem 2.24).

However, in general the global Hilbert function H_3 is not determined by the local Hilbert functions $\mathcal{O}_{p(k)}/J\mathcal{O}_{p(k)}$ at the points of its support — even when the support of \mathfrak{Z} is a single point! An exception is when $J\mathcal{O}_p$ is *conic*, a graded ideal in the local ring \mathcal{O}_p (Example 1.11). In this *conic* local case, we have $\Delta H_3 = H(\mathcal{O}_p/J\mathcal{O}_p)$, and the ideal I_3 and its global inverse system L_3 is easily read from J, L' ([IK, Lemma 6.1], Proposition 2.18).

How do we determine the global inverse system L_3 from the local inverse systems $L'(J(k))$? We answer by suitably homogenizing the local inverse systems (Definition 2.4). Our main result is that we determine *generators* of L_3 from generators of the local inverse systems: we give an algorithm to determine the Macaulay dual L_3 of the one-dimensional coordinate ring \mathcal{O}_3 directly from the local inverse systems (Lemma 2.9, Theorems 2.24, 2.29). In Section 2.1 we consider the case \mathfrak{Z} has support the coordinate point $p = p_0 = (0 : \dots : 0 : 1) \in \mathbb{P}^n$. Since $\mathcal{I}_p = J\mathcal{O}_p$ defines an Artinian quotient, we have $J \supset M'^{j+1}$, $M' = (y_1, \dots, y_n)$ for some integer $j > 0$, and we may replace \mathcal{O}_p by R' . Thus J has an inverse system $L'(J) \subset \Gamma'$ and there is no need to complete to $\hat{\Gamma}'$. However $L'(J)$ will usually not be graded. We define the homogenization of $L'(J)$ when $p = p_0$ in Definition 2.4, then show it is the same as L_3 and find suitable *generators* of L_3 in Lemmas 2.7, and 2.9.

We give examples that show a surprising behavior of this globalization with respect to the regularity degree $\sigma(\mathfrak{Z})$. The degree i component $(L_3)_i$ may not be determined by the local L'_i , and may not be the homogenization of $(L_3)_{\sigma(\mathfrak{Z})}$ to degree i (Examples 2.13 to 2.17), although the socle degree component $L_{\alpha(\mathfrak{Z})}$ does determine L_3 (Proposition 2.11(ii)).

In Section 2.2 we determine the global inverse system L_3 for a scheme concentrated at an arbitrary point $p \in \mathbb{A}^n \subset \mathbb{P}^n$. We then prove a projective space *Comparison Theorem* (Theorem 2.24) relating the inverse system at p to one concentrated at the origin. This result is different from our version of Macaulay's Comparison Lemma, which describes local inverse systems at points $p \in \mathbb{A}^n$, as the product of a local inverse system at the origin and an *exponential* power series f_p (Lemma 2.22). Rather, our *Comparison Theorem* shows that L_3 and its generators can be obtained from $L_{3'}$, the corresponding inverse system L_{p_0} at the origin p_0 of \mathbb{A}^n , by suitably substituting the divided powers of the linear form $L_p = \sum_k a_k X_k$ determined by the coordinates $(a_1 : \dots : a_n : 1)$ of p , for the powers of $Z = X_{n+1}$ in L_{p_0} .

We complete our study of globalization in Section 2.3. The Decomposition Theorem 2.29 handles the transition to arbitrary zero-dimensional schemes. We discuss regularity degree, giving an upper bound in terms of the invariant $\alpha(\mathfrak{Z})$ when the number of irreducible components of \mathfrak{Z} is less or equal $n + 2$ (Proposition 2.34).

In Section 3 we give the application that motivated our globalization of inverse systems. When \mathfrak{Z} is a locally Gorenstein punctual subscheme of \mathbb{P}^n , then $\text{Sym}(H_{\mathfrak{Z}}, j)$ is an upper bound for the Hilbert function $H(A)$ of a Gorenstein Artinian (GA) quotient A of $\mathcal{O}_{\mathfrak{Z}}$, having socle degree j . As a consequence of our construction of $L_{\mathfrak{Z}}$ we show that this upper bound is always achieved by some GA quotient of $\mathcal{O}_{\mathfrak{Z}}$, hence for almost all GA quotients of socle degree j , provided that j is sufficiently large (Theorem 3.3).

In an earlier related paper we showed that there is no level Artinian algebra A of Hilbert function $H(A) = (1, 3, 4, 5, \dots, 6, 2)$ [ChoI1]. This shows that Theorem 3.3 cannot be simply extended to a scheme \mathfrak{Z} that is locally of type two — having two-dimensional socle in a single degree. In a sequel paper [ChoI2] we will determine the global Hilbert function $H_{\mathfrak{Z}}$ for compressed Gorenstein subscheme $\mathfrak{Z} \subset \mathbb{P}^n$. Then, using Theorem 3.3, we will exhibit families $\mathbb{P}\mathbf{GOR}(T)$ of graded Gorenstein Artin algebras of embedding dimension r and certain Hilbert functions $T = H(s, j, r)$, $r \geq 5$, s large enough, that contain several irreducible components (Remark 3.24.)

1.1 Inverse systems and Gorenstein subschemes of \mathbb{P}^n .

Uses of inverse systems

Macaulay used his inverse systems, a version of the classical notion of apolarity, to develop a theory of primary decomposition of ideals [Mac]. Consider a subscheme $\mathfrak{Z} = \text{Spec}(\mathcal{O}_p/\mathcal{I}_p)$ of affine n -space concentrated at the point p of \mathbb{A}^n , whose maximal ideal is $m_p \subset R' = k[y_1, \dots, y_n]$, and local ring \mathcal{O}_p . We may write also $\mathfrak{Z} = \text{Spec}(A)$, $A = R'/I'$, where A has finite length, and where I' satisfies $m_p \supset I' \supset m_p^{\alpha+1}$ for some α . The affine inverse system $L(I') \subset \hat{\Gamma}'$ is a finite R' -module, isomorphic to the dualizing module $\Omega(A)$. The number of generators of the submodule $L(I') \subset \hat{\Gamma}'$ is the *type* of A , the vector space dimension of the *socle* $\text{Soc}(A) = (0 : m)$ (Definition 1.7). In particular when A is a Gorenstein Artin algebra — one whose socle is a vector space of dimension 1 — the local inverse system has a single generator, and was termed by Macaulay a *principal system* [Mac, §60].

A. Terracini translated questions about the Hilbert function of ideals of functions vanishing to specified order at a set of general enough points in \mathbb{P}^n — the *interpolation problem* — to questions concerning the Hilbert functions of ideals generated by powers of the corresponding linear forms. This translation has led to new insights, some still conjectural, when $n \geq 3$ [Ter1, Ter2, EhR, I4], and also contributed to the solution of a *Waring problem* for forms — whether a generic homogeneous form of degree j can be written as a sum of powers of linear forms. The answer followed from J. Alexander and A. Hirschowitz’s solution of the order-two interpolation problem ([Ter2, AlH, Cha2, I3], [IK, §2.1], [BrOt]). Principal inverse systems generated by certain forms associated to partitions, occurred as spaces of *harmonics* in the recent *n -factorial conjecture* in combinatorics and geometry [Ha]; they are also related to constant-coefficient partial differential equations [Rez]. Inverse systems have been studied further, sometimes as Matlis duality/injective envelope (see [No, NR], [BS, Chapter 10]). Related to the simpler Matlis duality are the deeper topics of dualizing modules, residues, and local cohomology [L-J, BS, Schz].

F.H.S. Macaulay introduced inverse systems in the context of affine space. To our knowledge, before 2000 when we submitted an earlier version, there had been no systematic study of inverse systems in the context of projective spaces, beyond the case of fat points considered by A. Terracini and others ([Ter1, Ter2, EhR, EmI, Ge, I3], see [Tes] for an exception). Since 2000 several other authors have explored this topic, notably M. Elkadi and B. Mourrain, and J. Brachat in his thesis who discuss generalized additive decompositions [ElMo, Bra]. There has been recent interest in non-homogeneous principal inverse systems in connection with the study of scheme or “cactus” length of forms, by K. Ranestad and F.-O. Schreyer [RS1, RS2] and others [BeRa, BuB].

D. M. Meyer and L. Smith develop the theory of Poincaré duality algebras — the Gorenstein case of inverse systems — in the language of Hopf algebras. They construct new Gorenstein algebras

from existing ones by using their corresponding principal inverse systems [MeSm]. L. Smith and R. E. Stong consider a pair of Poincaré duality algebras

$$A = k[x_1, x_2, \dots, x_n]/J \mapsto k[x_1, x_2, \dots, x_n, x_{n+1}]/I = B,$$

where B is a free A -module and f_J, f_I are generators of principal inverse systems of J and I , respectively. They obtain f_I from f_J by means of a homogenization process [SmSt, Theorem 2.5] and they make new Poincaré duality algebras from a given A [SmSt, Corollary 3.5].

A zero-dimensional scheme $\mathfrak{Z} \subset \mathbb{P}^n$ is the union of a finite number of schemes $\mathfrak{Z}(i)$, each supported at a single point $p(i) \in \mathbb{P}^n$. So we may define dualizing modules $D(V)$ for B -modules V , where $B = R/I_{\mathfrak{Z}}$, as direct sums of the dualizing modules at the finite number of points: thus $D(V) = \text{Hom}(V, \bigoplus E(i))$, where $E(i) = E(R/M(i))$ is the injective hull of the residue field k of B at the maximal ideal $M(i)$ at the i -th point $p(i)$ of the support. This viewpoint is adopted by Curtis-Reiner [CR, p.37], and was used by R. Michler in [Mi]. However, our task is in one sense easier, and in another harder. Easier, since our ideal $I_{\mathfrak{Z}}(i)$ includes a power of the maximal ideal $m_{p(i)}$ at $p(i)$ so we may avoid the full injective hull and deal locally with *dual polynomials*, or dual polynomials times an exponential ([L-J, Mac], Lemma 2.22, Remark 2.23). Harder since we wish here to consider a global inverse system for $R/I_{\mathfrak{Z}}$ that is embedded in Γ , rather than simply being an R -module. We pass from the local inverse systems for the ideals $I_{\mathfrak{Z}(i)}$ at each point, finite submodules of $\widehat{\Gamma}'$, to the global inverse system for the ideal $I_{\mathfrak{Z}} \subset R$, which is not finitely generated, but is determined by a finite number of its elements.

Gorenstein Artin (GA) algebras are minimal reductions of Gorenstein algebras. Gorenstein algebras are a natural generalization of complete intersections. Artin algebras, and in particular GA algebras occur in the study of mapping germs of differentiable maps. Recently a category of commutative Frobenius algebras, that correspond to non-graded Gorenstein Artin algebras have been identified with the category of two-dimensional topological quantum field theories [Ab].

J. Watanabe showed that the family $\text{ZGOR}(T)$ of all, not necessarily graded standard GA algebras having a *symmetric* Hilbert function T is fibred over the family $\mathbb{P}\text{GOR}(T)$ parametrizing graded GA algebras of Hilbert function T by the map $A \rightarrow Gr_m(A)$ to the associated graded algebra ([Wa], [I2, Prop. 1.7]). Our work here relates to the component structure of $\mathbb{P}\text{GOR}(T)$ and we hope there could be application to these other fields.

The inverse system viewpoint can be used to parametrize Gorenstein Artin algebra quotients of R' having a given Hilbert function [I2]). Several authors have studied from this or related viewpoints *compressed algebras* — those having a maximum possible Hilbert function, given the socle degree and embedding dimension (see [I1, FL, Bo2, Za]). See also the “points épais” discussion by J. Emsalem [Em] and the foundational study of D. Laksov [La].

Main Results and Applications.

We first translate into the language of global inverse systems, some basic algebraic properties of the coordinate ring $R/I_{\mathfrak{Z}}$, where $I_{\mathfrak{Z}}$ is the defining ideal of a zero-dimensional subscheme of \mathbb{P}^n . We consider such properties as “there is a linear non-zero divisor ℓ on $R/I_{\mathfrak{Z}}$ ”, and the *type* of $R/I_{\mathfrak{Z}}$. We then use the inverse systems to study such questions as “When is \mathfrak{Z} arithmetically Gorenstein (aG)?”, and “When can $I_{\mathfrak{Z}}$ be recovered from a general form F annihilated by $I_{\mathfrak{Z}}$?” (F must have sufficiently high degree). We discuss “When is \mathfrak{Z} aG?” in Example 2.14, Proposition 2.18, Corollary 2.20, Remark 2.31, and in Examples 2.32, 3.16, 3.17. As to the latter question, it is not hard to see that if \mathfrak{Z} is Gorenstein and is also either smooth, or concentrated at a single point and *conic* — defined by a homogeneous ideal \mathcal{I}_p of the local ring \mathcal{O}_p — then we can recover $I_{\mathfrak{Z}}$ from F (see [Bo2], [IK, Lemma 6.1]). However, it is also easy to see that the second order neighborhood of a point $p \in \mathbb{P}^n, n \geq 2$, a non-Gorenstein scheme defined by m_p^2 , cannot be recovered in this manner (Example 3.1). When can we recover \mathfrak{Z} from an Artinian Gorenstein quotient?

We answer this question in Theorem 3.3. Given a positive integer j and a sequence $H_{\mathfrak{Z}}$, we let $\text{Sym}(H_{\mathfrak{Z}}, j)$ be the sequence

$$\text{Sym}(H_{\mathfrak{Z}}, j)_i = \begin{cases} (H_{\mathfrak{Z}})_i, & \text{if } i \leq j/2; \\ (H_{\mathfrak{Z}})_{j-i}, & \text{if } i \geq j/2. \end{cases} \quad (1.1)$$

We denote by $\sigma(\mathfrak{Z})$ the *Castelnuovo – Mumford regularity* of \mathfrak{Z} , we set $\tau(\mathfrak{Z}) = \sigma(\mathfrak{Z}) - 1$, and let $\alpha(\mathfrak{Z})$ be the *maximum socle degree* of the local coordinate ring of any irreducible component $\mathfrak{Z}(i)$ (see Definition 2.3). We let $\beta(\mathfrak{Z}) = \tau(\mathfrak{Z}) + \max\{\tau(\mathfrak{Z}), \alpha(\mathfrak{Z})\}$, $I_{\mathfrak{Z}}$ be the defining ideal, $L_{\mathfrak{Z}}$ its inverse system, and now state our main result, Theorem 3.3.

Theorem. RECOVERING THE SCHEME \mathfrak{Z} FROM A GORENSTEIN ARTIN QUOTIENT. *Let \mathfrak{Z} be a locally Gorenstein zero-dimensional subscheme of \mathbb{P}^n over an algebraically closed field \mathbf{k} , $\text{char } \mathbf{k} = 0$ or $\text{char } \mathbf{k} > j$, and let $L_{\mathfrak{Z}} = (I_{\mathfrak{Z}})^{-1}$. Then we have*

- (i) *If $j \geq \beta(\mathfrak{Z})$, and F is a general enough element of $(L_{\mathfrak{Z}})_j$, then $H(R/\text{Ann}(F)) = \text{Sym}(H_{\mathfrak{Z}}, j)$.*
- (ii) *If $j \geq \beta(\mathfrak{Z})$, and F is a general enough element of $(L_{\mathfrak{Z}})_j$, then for i satisfying $\tau(\mathfrak{Z}) \leq i \leq j - \alpha(\mathfrak{Z})$ we have $\text{Ann}(F)_i = (I_{\mathfrak{Z}})_i$. Equivalently, we have $R_{j-i} \circ F = (L_{\mathfrak{Z}})_i$.*
- (iii) *If $j \geq \max\{\beta(\mathfrak{Z}), 2\tau(\mathfrak{Z}) + 1\}$, and $F \in (L_{\mathfrak{Z}})_j$ is general enough, then $\text{Ann}(F)$ determines \mathfrak{Z} uniquely. If $I_{\mathfrak{Z}}$ is generated in degree $\tau(\mathfrak{Z})$, then $j \geq \max\{\beta(\mathfrak{Z}), 2\tau(\mathfrak{Z})\}$ suffices.*

Thus, we may recover \mathfrak{Z} from a general dual form F when \mathfrak{Z} is locally Gorenstein and j is large enough. The authors show elsewhere that Theorem 3.3 does not extend simply to subschemes \mathfrak{Z} that are not Gorenstein, by showing that the sequence $H = (1, 3, 4, 5, \dots, 6, 2)$ cannot occur as the Hilbert function of a level algebra — one having socle in a single degree ([ChoI1]).

The question of which symmetric sequences T of integers are Gorenstein sequences — Hilbert functions of a graded Gorenstein Artin algebra — is open in embedding dimension $r \geq 4$. We do not here find any new Gorenstein sequences of the form $T = \text{Sym}(H_{\mathfrak{Z}}, j)$: each sequence $H_{\mathfrak{Z}}$ already occurs for a smooth scheme \mathfrak{Z} by [Mar] (see Theorem 1.12), and Theorem 3.3 was already known for smooth schemes [Bo2], [IK, Theorem 5.3E, Lemma 6.1]. So an application of Theorem 3.3 is to relate the postulation punctual Hilbert scheme $\text{Hilb}_{\text{Gor}, H}^s(\mathbb{P}^n)$ parametrizing degree- s Gorenstein subschemes $\mathfrak{Z} \subset \mathbb{P}^n$, satisfying $H_{\mathfrak{Z}} = H$, with the scheme $\text{PGOR}(T)$ parametrizing graded Gorenstein Artin algebras of Hilbert function $T = \text{Sym}(H, j)$. In this direction see also [K11, K12].

In Section 3.3 we explore a second viewpoint on our construction of a global inverse system from the local inverse system: we discuss *generalized additive decomposition* of a form F when $r > 2$. A binary form F of degree j , always has a length- s generalized additive decomposition (GAD), with $s \leq (j+2)/2$: this is either a sum of j -th powers of s distinct linear forms, or a sum

$$F = \sum_i B_i L_i^{[j+1-s_i]}, \deg B_i = s_i - 1, \deg L_i = 1, s = \sum s_i. \quad (1.2)$$

The existence of such an additive decomposition when $r = 2$ is equivalent to there being a form $h \in \text{Ann}(F)$ that can be written $h = \prod_i \ell_i^{s_i}$, where $\ell_i \circ L_i = 0$. Thus, the additive decomposition of equation (1.2) corresponds to a zero-dimensional scheme $\mathfrak{Z} : h = 0 \subset \mathbb{P}^1$, whose irreducible components $\mathfrak{Z}_i : \ell_i^{s_i} = 0$ have specified multiplicities s_i . If $2s \leq j+1$ then it is classical that the GAD as in equation (1.2) is unique: see [IK, §1.3, Proposition 1.36, Theorem 1.43]. For any embedding dimension, we say that a zero-dimensional scheme $\mathfrak{Z} \subset \mathbb{P}^n$ is an *annihilating scheme* of the form $F \in R$, if $I_{\mathfrak{Z}} \circ F = 0$. Since for a zero-dimensional scheme, $I_{\mathfrak{Z}} = \cap I_{\mathfrak{Z}_i}$, where \mathfrak{Z}_i are the irreducible components of \mathfrak{Z} , and we have $I_{\mathfrak{Z}}^\perp = \sum_i I_{\mathfrak{Z}_i}^\perp$, it follows that any form F annihilated by \mathfrak{Z} can be written as a *generalized sum*, a sum of forms annihilated by

the components \mathfrak{Z}_i . In determining very concretely the inverse systems of $I_{\mathfrak{Z}_i}$, we are partially answering the question, what is a generalized additive decomposition? In particular, when $r = 3$, many forms F have a *tight* annihilating scheme $\mathfrak{Z}_F \subset \mathbb{P}^2$ that is unique. As well, there is often a unique *generalized additive decomposition*, up to trivial multiplications. This occurs when the Hilbert function $H(R/\text{Ann}(F))$ contains as a consecutive subsequence (s, s, s) . Then, there is a unique degree- s annihilating scheme according to [IK, Theorem 5.31], and, as we shall see, a corresponding unique *generalized additive decomposition* for F (Theorems 3.21 and 3.22). Some results about uniqueness of GAD in a quite different language have occurred since our original submission in the thesis of J. Brachat [Bra, §4].

1.2 Notation and basic facts.

We will assume throughout, unless specifically stated otherwise, that the base field \mathbf{k} satisfies $\text{char } \mathbf{k} = 0$, or $\text{char } \mathbf{k} = p > j$, where j is the maximum degree of any form considered (see Example 2.2 for the necessity of this assumption). We will also assume either that \mathbf{k} is algebraically closed, or that all zero-dimensional schemes considered have as support \mathbf{k} -rational points. Let C denote the \mathbf{k} -vector space $\langle x_1, \dots, x_{n+1} \rangle$, and $C^* = \langle X_1, \dots, X_{n+1} \rangle$ denote its dual; recall that the divided power ring $\Gamma = \Gamma(C^*) = \mathbf{k}_{DP}[X_1, \dots, X_{n+1}]$ satisfies,

$$\Gamma = \bigoplus \Gamma_j = \bigoplus \text{Hom}(R_j, \mathbf{k}), \text{ with } \Gamma_j = \langle \{X^{[U]} \mid |U| = j\} \rangle,$$

the span of the dual generators to $x^U \in R$, where here U denotes the multiindex $U = (u_0, \dots, u_n)$, of length $|U| = \sum u_i$. For convenience we set $X^{[U]} = 0$ if any component of U is negative. The multiplication in Γ is defined by

$$X^{[U]} \cdot X^{[V]} = \binom{U+V}{U} X^{[U+V]}. \quad (1.3)$$

We denote by R', Γ' , respectively, the corresponding rings $R' = \mathbf{k}[y_1, \dots, y_n]$ and $\Gamma' = \mathbf{k}_{DP}[Y_1, \dots, Y_n]$, respectively. We have $\Gamma' \cong E' = \text{Hom}_{R'}(R', R'/M'), M' = (y_1, \dots, y_n)$. We denote by $\widehat{\Gamma}'$ the completion of Γ' with respect to M' ; thus, $\widehat{\Gamma}'$ is a divided power series ring. The rings R', Γ' correspond to the point $p_0 = (0 : \dots : 0 : 1)$ in \mathbb{P}^n , whose maximal ideal is $m_{p_0} = (x_1, \dots, x_n) \subset R$. We recall below the contraction action of $R = \mathbf{k}[x_1, \dots, x_{n+1}]$ on the divided power ring. Note that, given our assumption excluding low characteristics, each theorem about inverse systems stated in the context of the contraction action of R on Γ , has an analogue for the partial differential operator (PDO) action of R on $\mathcal{R} = \mathbf{k}[X_1, \dots, X_{n+1}]$, a second copy of the polynomial ring. When $\text{char } \mathbf{k} = 0$, there is a natural GL -invariant homomorphism $\phi : \mathcal{R} \rightarrow \Gamma$, $\phi(X^U) = U! X^{[U]} \in \Gamma$ [IK, Appendix A]. To keep the exposition simple, we will in general restrict ourselves to the contraction action. Note that we use here a different notation than Macaulay's $\mathbf{k}[x_1^{-1}, \dots, x_{n+1}^{-1}]$ for the injective envelope $E = E(\mathbf{k})$ ([Mac, MeSm], [Ei, Theorem 21.6]). The claims implicit in (iv),(v) of the following Definition are shown in Lemmas 1.4 and 1.6 below.

Definition 1.1. Inverse Systems

(i) (*contraction action*) If $h = \sum a_K x^K \in R, F = \sum b_U X^{[U]} \in \Gamma$, then

$$h \circ F = \sum_{K,U} a_K b_U X^{[U-K]}.$$

(ii) (*partial differentiation action* — PDO) If $h \in R, F \in \mathcal{R} = \mathbf{k}[X_1, \dots, X_{n+1}]$, then

$$h \circ F = h(\partial / \partial X_1, \dots, \partial / \partial X_{n+1})(F) \in \mathcal{R}.$$

- (iii) A *homogeneous inverse system* $W \subset \Gamma$ is a graded R -submodule of Γ under the contraction action. Thus $W = W_0 \oplus \cdots \oplus W_j \oplus \cdots \subset \Gamma$ is an inverse system if and only if $\forall i \leq j, R_i \circ W_j \subset W_{j-i}$.
- (iv) (*inverse system of a graded ideal*) If I is a graded ideal of R , we will denote by I^{-1} or by I^\perp the homogeneous inverse system of I , namely the R -submodule of Γ given by $I^\perp = \bigoplus I_j^\perp$, where

$$I_j^\perp = \{F \in \Gamma_j \mid h \circ F = 0 \quad \forall h \in I_j\}.$$

- (v) (*ideal of an inverse system*) If $W \subset \Gamma$ is an inverse system, then we denote by I_W the ideal $I_W = \text{Ann}(W)$ where $(I_W)_j = \{h \in R_j \mid h \circ w = 0, \quad \forall w \in W\}$.
- (vi) (*local inverse system*) An inverse system in $\widehat{\Gamma}'$ is an R' -submodule of $\widehat{\Gamma}'$ under the contraction action. If J is any ideal of $R' = \mathbf{k}[y_1, \dots, y_n]$, then we denote by $J^\perp = J^{-1} \in \widehat{\Gamma}'$, the inverse system of all elements of $\widehat{\Gamma}'$ annihilated by J , in the contraction action. The ideal $I_W \subset R'$ of an inverse system $W \subset \widehat{\Gamma}'$ is the annihilator of W under the contraction action. [Warning: in general neither J nor J^\perp is homogeneous].

Henceforth in this paper, inverse systems in Γ (but not in $\Gamma', \widehat{\Gamma}'$) are assumed to be homogeneous. We will later need that the elements of R_1 act as differentials on Γ (Lemma 2.22).

Lemma 1.2. *If ℓ is an element of R_1 , and $F, G \in \Gamma_u, \Gamma_v$, respectively, then*

$$\ell \circ (F \cdot G) = (\ell \circ F) \cdot G + F \cdot (\ell \circ G). \quad (1.4)$$

Proof. By bilinearity, it suffices to show (1.4) when ℓ is a variable, and F, G are monomials, whence it suffices to show it when $R = \mathbf{k}[x], \Gamma = \mathbf{k}[X]$ in a single variable, and for $\ell = x, F = X^{[a]}, G = X^{[b]}$. There, it results from the definition of the multiplication in the divided power ring Γ , and the usual Pascal triangle binomial identity. \square

We need a simple result relating inverse systems and ideals. First we recall

- Definition 1.3.** (i) If $V \subset R_j$, and $i \geq 0$, we have $R_i \cdot V = \langle hv \mid h \in R_i, v \in V \rangle$; if also $i \leq j$ we have $(V : R_i) = \langle h \in R_{j-i} \mid R_i h \subset V \rangle$.
- (ii) If $W \subset \Gamma_j$ and $i \geq 0$, we have $R_{-i} \circ W =_{\text{def}} (W : R_i) = \langle \{F \in \Gamma_{j+i} \mid R_i \circ F \subset W\} \rangle$. If $W \subset \Gamma_j$ and $0 \leq i \leq j$ we have $R_i \circ W = \langle h \circ w \mid h \in R_i, w \in W \rangle$.

Lemma 1.4. INVERSE SYSTEM AND MATLIS DUALITY. *Assume that (V, W) is a pair of vector spaces satisfying $V \subset R_j$, $W \subset \Gamma_j$ and $V^\perp \cap \Gamma_j = W$. Then*

- (i) *If $0 \leq i$, then $(R_i \cdot V)^\perp \cap \Gamma_{j+i} = W : R_i$.*
- (ii) *If $0 \leq i \leq j$, then $(V : R_i)^\perp \cap \Gamma_{j-i} = R_i \circ W$.*
- (iii) *If $L \subset \Gamma$ is a homogeneous inverse system, then $\text{Ann}(L) \subset R$ is a graded ideal of R ; if I is a graded ideal of R , then $I^{-1} \subset \Gamma$ is a homogeneous inverse system. Furthermore, $\text{Ann}(L)^{-1} = L$; and $\text{Ann}(I^{-1}) = I$. Also $I^{-1} \cong \text{Hom}_{\mathbf{k}}(R/I, \mathbf{k})$, the Matlis dual of R/I .*
- (iv) *If the inverse system $L' \subset \Gamma'$ (not necessarily graded) has finite dimension as \mathbf{k} -vector space, then $I' = \text{Ann}(L')$ is an M' -primary ideal of R' , where $M' = (y_1, \dots, y_n)$. Conversely, an M' -primary ideal I' of R' determines a finite-dimensional inverse system of $L(I') \subset \Gamma'$.*

- (v) If $I' \subset R'$ is an ideal of finite colength c , defining an Artin quotient R'/I' with s distinct maximal ideals, then $I'^{-1} \subset \hat{\Gamma}'$ is a dimension- c inverse system of the form $I'^{-1} = \bigoplus_1^s L'(i)$, $L'(i) = V'(i)f_{p(i)}$, where $V'(i) \subset \Gamma'$ is a finite inverse system, and $f_{p(i)}$ is a specific power series (see (2.17)).

Proof. For (i), note that $(R_i \cdot V) \circ F = 0$ if and only if $V \circ (R_i \circ F) = 0$. And the last equality is equivalent to $R_i \circ F \subset W$.

For (ii), note that for any $h \in R_{j-i}$, $h \circ (R_i \circ W) = 0$ if and only if $R_i \circ (h \circ W) = 0$. And the last equality is equivalent to $R_i h \subset V$.

For (iii), note that I is a graded ideal of R if for each pair of non-negative integers (i, j) , $R_i \cdot I_j \subset I_{i+j}$, or, equivalently, if $(I_{i+j} : R_i) \supset I_j$. By (ii) the latter is equivalent to $R_i \circ \langle I_{i+j}^{-1} \rangle \subset I_j^{-1}$, implying that I^{-1} is a homogeneous inverse system. One shows similarly that the annihilator $\text{Ann}(L) \subset R$ of a homogeneous inverse system L is an ideal, using (i). That the double duals are the identities in this case follows from the exactness of the pairing $R_i \circ \Gamma_i \rightarrow k$. For (iv), note that if $L' \subset \Gamma'$ is finite dimensional then $L' \subset \Gamma'_{\leq j}$ for some integer j , hence $\text{Ann}(L') \supset M'^{j+1}$, and conversely. For (v), note that since $I' = \cap I'(i)$, the inverse system I'^{-1} is the direct sum of the inverse systems $L'(i)$ of the components $I(i)$ at the points $p(i)$ of support. Then use Lemmas 2.21 and 2.22 below. \square

Usually, a homogeneous inverse system $W \subset \Gamma$ is not finitely generated. In fact, if W is finitely generated, then $\dim_k W$ is finite, and by Lemma 1.4(iv) W determines an Artin algebra $A_W = R/I$, $I = \text{Ann}(W)$ with I an $M = (x_1, \dots, x_{n+1})$ -primary ideal. Recall

Definition 1.5. A graded ideal $I \subset R$ is *saturated* if it has no irreducible component primary to the irrelevant ideal M , equivalently, if $I = I : M^\infty = \{f \mid \exists k \geq 0, M^k \cdot f \subset I\}$. This is equivalent to,

$$\forall a, b \in \mathbb{N}, a \leq b \quad I_a = (I_b : R_{b-a}) = \{f \in R_a \mid R_{b-a} \cdot f \subset I_b\}. \quad (1.5)$$

If $\dim(R/I) = 1$, I is saturated if and only if there is a linear non-zero divisor for R/I in R .

Note that the condition of equation (1.5) results from the more usual saturation condition,

$$\exists N \in \mathbb{N} \mid \forall a, \forall b \geq \max(N, a), \quad I_a = (I_b : R_{b-a}). \quad (1.6)$$

Lemma 1.6. MACAULAY'S CORRESPONDENCE. *There is a one-to-one correspondence between homogeneous inverse systems $W \subset \Gamma$ and graded ideals I of R , given by $I \mapsto I^{-1} \subset \Gamma$, and $W \mapsto I_W = \text{Ann}(W) \subset R$. The ideal I_W is saturated if and only if the inverse system W satisfies*

$$\forall a, b \in \mathbb{N}, a \leq b \quad W_a = R_{b-a} \circ W_b. \quad (1.7)$$

Furthermore, the element $\ell \in R_i$ is a non-zero divisor for R/I if and only if $W = I^{-1}$ satisfies

$$\forall b \in \mathbb{N}, b \geq i, \quad \text{we have } \ell \circ W_b = W_{b-i}. \quad (1.8)$$

Proof. The 1-1 correspondence has been shown in Lemma 1.4. The relation (1.7) follows from (1.5), using Lemma 1.4(ii). That ℓ is a non-zero divisor for R/I is equivalent to

$$\text{for each integer } b \geq i, \text{ and } \forall h \in R_{b-i}, \ell \cdot h \in I_b \text{ implies that } h \in I_{b-i}. \quad (1.9)$$

Letting $W = I^{-1}$, we may translate the implication in (1.9) equivalently as followings:

$$\begin{aligned} (\ell \cdot h) \circ W_b &= 0 \text{ implies that } h \circ W_{b-i} = 0, \\ h \circ (\ell \circ W_b) &= 0 \text{ implies that } h \circ W_{b-i} = 0, \\ (\ell \circ W_b)^\perp \cap R_{b-i} &\subset (W_{b-i})^\perp \cap R_{b-i}, \\ \ell \circ W_b &\supset W_{b-i}. \end{aligned}$$

Since by definition $R_i \circ W_b \subset W_{b-i}$, this shows the criterion (1.8). \square

We will term an inverse system W of Γ *saturated* if W satisfies (1.7) ; that is, W arises from a saturated ideal. We now recall the definitions of socle, and type.

Definition 1.7. (i) Let A be a local ring with the maximal ideal m . Then the *socle* of A , $\text{Soc}(A)$, is defined as $(0 : m) \subset A$. Further, if A is an Artin algebra, the *type* of A is the dimension $\dim_k \text{Soc}(A)$, and the socle degree is the maximum degree i in which $\text{Soc}(A)_i$ is non-zero.
(ii) If $\mathfrak{Z} \subset \mathbb{P}^n$ is a zero-dimensional scheme, and $I_{\mathfrak{Z}}$ is a saturated ideal defining \mathfrak{Z} , then the type of $\mathcal{O}_{\mathfrak{Z}} = R/I_{\mathfrak{Z}}$ is defined as $\dim_k(\mathcal{O}_{\mathfrak{Z}}/\ell\mathcal{O}_{\mathfrak{Z}})$. Here, ℓ is a linear non-zero divisor of $\mathcal{O}_{\mathfrak{Z}}$ and $\mathcal{O}_{\mathfrak{Z}}/\ell\mathcal{O}_{\mathfrak{Z}}$ is the Artin local ring with the maximal ideal $(x_1, \dots, x_{n+1})/(I_{\mathfrak{Z}}, \ell)$.

It is well known that this notion of type does not depend on the non-zero divisor ℓ used: the type is the rank of the last module in a free R -resolution of A , and these ranks remain the same when we quotient by any non-zero divisor. See [BH, Lemma 1.2.19] for the analogue in the case B is local of arbitrary dimension.

Corollary 1.8. *Suppose $I \subset R$ has inverse system $W \subset \Gamma$. The vector space $I_j/\langle R_1 \cdot I_{j-1} \rangle$ of degree- j generators of I is dual to the vector space $\langle W_{j-1} : R_1 \rangle/W_j$. The vector space $(I_{j+1} : R_1)/I_j$ of degree- j socle elements of $A = R/I$ is dual to the vector space W_j/R_1W_{j+1} of degree- j generators of W .*

Proof. This is immediate from Lemma 1.6 and Lemma 1.4 (i),(ii). \square

We now show how to recognize the type of I from the inverse system; we then describe the inverse system of the projective closure of a scheme. We will complete our listing of basic facts by characterizing the ideals defining zero-dimensional schemes and their inverse systems (Theorem 1.12, Proposition 1.13).

Lemma 1.9. *Let $I = I_{\mathfrak{Z}}$ be the homogeneous saturated ideal defining a zero-dimensional subscheme $\mathfrak{Z} \subset \mathbb{P}^n$, let $W = I^{-1} \subset \Gamma$ be the inverse system of I . Let $\ell \in R_1$ be a non-zero divisor for $B = R/I$, set $A = B/\ell B$ with maximal ideal m . Denote by $\Gamma_{\ell} = \ell^{\perp} \subset \Gamma$ the R -submodule of Γ perpendicular to ℓ , and let $W_{\ell} = W \cap \Gamma_{\ell}$. Then*

- (i) W_{ℓ} is the dual module of A .
- (ii) $W_{\ell}/\langle M \circ W_{\ell} \rangle \cong (\text{Soc}(A))^{\vee}$, the dual space to $\text{Soc}(A)$.

Proof. Since $A = B/\ell B$ is isomorphic to $R/(I, \ell)$, its dual module is the inverse system of (I, ℓ) , so $A^{\vee} \cong I^{-1} \cap \ell^{\perp} = W_{\ell}$: this shows (i). Also, $((I, \ell) : M)$ is perpendicular to $M \circ W_{\ell}$. Thus, we have $A = R/(I, \ell)$ and $\text{Soc}(A) = (0 : m) = ((I, \ell) : M)/(I, \ell)$, hence

$$(\text{Soc}(A))^{\vee} = (R/(I, \ell))^{\vee}/(R/((I, \ell) : M))^{\vee} = (I, \ell)^{\perp}/((I, \ell) : M)^{\perp} \cong W_{\ell}/\langle M \circ W_{\ell} \rangle.$$

This is (ii), and completes the proof. \square

When \mathfrak{Z} is a zero-dimensional scheme of $\mathbb{A}^n \subset \mathbb{P}^n$, its projective closure has an empty intersection with the hyperplane at infinity: $z = 0$, since \mathfrak{Z} is already closed. However, in fact there is a graded Artinian algebra $R'/(I_{\mathfrak{Z}})_{z=0}$ lying on the hyperplane at infinity, uniquely determined by \mathfrak{Z} , and whose Hilbert function determines $H(R/I_{\mathfrak{Z}})$. We also show the connection with the global inverse system. Recall that the Hilbert function $H(B)$ for an R -module B is the sequence $H(B)_i = \dim_k B_i$, with B_i the degree- i component of the associated graded module $\text{Gr}_M(B)$. We will write Hilbert functions of submodules of Γ in the order of increasing degrees, so that $H(\Gamma) = H(R)$. We define the sequence ΔH by $\Delta H_i = H_i - H_{i-1}$.

Lemma 1.10. PROJECTIVE CLOSURE. When $R = \mathbb{k}[x_1, \dots, x_n, z]$ and $\ell = z$ then $\Gamma_z = z^\perp = \mathbb{k}_{DP}[X_1, \dots, X_n]$. Suppose that \mathfrak{Z} is a zero-dimensional scheme of $\mathbb{A}^n : z = 1$, with global inverse system $W = L_{\mathfrak{Z}}$. Then z is a non-zero divisor for $R/I_{\mathfrak{Z}}$, and $W_z = W \cap \Gamma_z$ satisfies

- (i) $\Delta H(R/I_{\mathfrak{Z}}) = H(R/(I_{\mathfrak{Z}}, z)) = H(W_z)$.
- (ii) There is an exact sequence, $0 \rightarrow W_z(i) \rightarrow W(i) \xrightarrow{z \circ} W(i-1) \rightarrow 0$, where the homomorphism $z \circ : W(i) \rightarrow W(i-1)$ is the contraction action of $z \in R$ on Γ as in Definition 1.1(i).
- (iii) The above sequence is dual to $0 \rightarrow (R/I_{\mathfrak{Z}})(i-1) \xrightarrow{m_z} (R/I_{\mathfrak{Z}})(i) \rightarrow \mathbb{k}[x_1, \dots, x_n]/(I_{\mathfrak{Z}})_{z=0} \rightarrow 0$ where the homomorphism m_z is multiplication by z .

In (i), (ii) above, z, W_z may be replaced by ℓ, W_ℓ , when \mathfrak{Z} is an arbitrary zero-dimensional subscheme of \mathbb{P}^n , provided ℓ is a non-zero divisor for $R/I_{\mathfrak{Z}}$.

Proof. If z were a zero divisor for $R/I_{\mathfrak{Z}}$, then z would be contained in an associated prime of $I_{\mathfrak{Z}}$, contradicting the assumption $\mathfrak{Z} \subset \mathbb{A}^n$. (iii) is immediate, That z is a non-zero divisor implies (iii). The statement (ii) follows from (iii) by dualizing, and (i) follows from these exact sequences by taking vector space dimensions. \square

Example 1.11. Let $I_{\mathfrak{Z}} = (xy, x^2z - y^3, x^3) \subset R = \mathbb{k}[x, y, z]$; then \mathfrak{Z} is a degree-5 scheme concentrated at the point $p_0 = (0 : 0 : 1)$ of \mathbb{P}^2 (the origin of \mathbb{A}^2), having global Hilbert function $H_{\mathfrak{Z}} = H(R/I_{\mathfrak{Z}}) = (1, 3, 5, 5, \dots)$. The Artin algebra $A = R/(I_{\mathfrak{Z}}, z) \cong \mathbb{k}[x, y]/(xy, x^3, y^3)$ has Hilbert function $H(A) = \Delta H_{\mathfrak{Z}} = (1, 2, 2, 0)$, and is the *boundary* of \mathfrak{Z} on the line at infinity: $z = 0$. The inverse system $W = (I_{\mathfrak{Z}})^{-1} \subset \Gamma = \mathbb{k}_{DP}[X, Y, Z]$ satisfies

$$\begin{aligned} W_3 &= \langle X^{[2]}Z + Y^{[3]}, Y^{[2]}Z, YZ^{[2]}, XZ^{[2]}, Z^{[3]} \rangle \\ W_2 &= \langle X^{[2]}, Y^{[2]}, YZ, XZ, X^{[2]} \rangle \\ W_1 &= \langle X, Y, Z \rangle; \quad W_0 = \langle 1 \rangle, \end{aligned}$$

and $W_z = \langle 1, X, Y, X^{[2]}, Y^{[2]} \rangle = W \cap \Gamma_z = W \cap \mathbb{k}_{DP}[X, Y] \subset \Gamma$ is the dual module to A .

When we consider $\mathfrak{Z} \subset \mathbb{A}^2$, by setting $z = 1$ in $I_{\mathfrak{Z}}$, we find $I' = (xy, x^2 - y^3)$, which defines a scheme concentrated at p_0 of local Hilbert function $H' = (1, 2, 1, 1)$, different from $\Delta H_{\mathfrak{Z}}$.

If we consider instead \mathfrak{Z}' , defined by (x^2, xy, y^4) , we would find the same local Hilbert function H' for \mathfrak{Z}' , but now $H_{\mathfrak{Z}'} = (1, 3, 4, 5, \dots)$, the sum function, since \mathfrak{Z}' is *conic*. This example shows that the local Hilbert function H' does not determine the global Hilbert function $H_{\mathfrak{Z}}$.

We recall next a well known result, see for example [GeM, Mar, Or]. We quote most of it from [IK, Theorem 1.69]. A scheme $\mathfrak{Z} \subset \mathbb{P}^n$ is *arithmetically Cohen-Macaulay* if $R/I_{\mathfrak{Z}}$ is Cohen-Macaulay; if $\dim \mathfrak{Z} = 0$, this is equivalent to there being a non-zero divisor in R for $R/I_{\mathfrak{Z}}$. Recall that $\tau(\mathfrak{Z}) = \min\{i \mid \dim_{\mathbb{k}}((R/I_{\mathfrak{Z}})_i) = s\}$.

Theorem 1.12. PUNCTUAL SCHEMES. Let \mathfrak{Z} be a degree- s zero-dimensional subscheme of \mathbb{P}^n , and let $I = I_{\mathfrak{Z}}$ be its saturated defining ideal. Then \mathfrak{Z} is arithmetically Cohen-Macaulay, and

- (i) The Hilbert function $H(R/I)$ is nondecreasing in i , and stabilizes at the value s for $i \geq \tau(\mathfrak{Z})$. We have $\tau(\mathfrak{Z}) \leq s - 1$, with equality if and only if \mathfrak{Z} is contained in a line.
- (ii) The Castelnuovo-Mumford regularity $\sigma = \sigma(\mathfrak{Z})$ satisfies $\sigma = \tau(I) + 1$. In particular, if $i \geq \sigma$, then $I_i = R_{i-\sigma} \cdot I_\sigma$. Thus, I is generated by degree σ .
- (iii) The first difference $\Delta(H(R/I)) = C = (1, c_1, \dots, c_\tau, 0)$ is an O -sequence (the Hilbert function of some Artin quotient of R'), with $s = \sum c_i$.

- (iv) Every O -sequence $C = (1, c_1, \dots, c_\tau, 0)$, $c_1 \leq n$, $c_\tau \neq 0$, $\sum c_i = s$, occurs as $\Delta H(R/I)$ for some degree- s zero-dimensional scheme \mathfrak{Z} with $\tau(\mathfrak{Z}) = \tau$, consisting of smooth points.

Conversely, any saturated ideal $I \subset R$ satisfying the Hilbert function conditions (i), (iii) above for $H(R/I)$ is the defining ideal of such a zero-dimensional subscheme, namely $\mathfrak{Z} = \text{Proj } (R/I) \subset \mathbb{P}^n$.

Proof outline. There are direct proofs of (i)–(iii) in [Or, GeM]; see also [IK, Theorem 1.69]. Let I be an ideal of R , such that R/I has dimension one. One can show cohomologically that I saturated is equivalent to R/I being Cohen-Macaulay (see, for example, [IK, Lemma 1.67]), and this is equivalent to a general element ℓ of R_1 being a non-zero divisor for R/I . Then ΔH is the Hilbert function of $R/(I_3, \ell)$, so is an O -sequence. That $\sigma = \tau + 1$ is the Castelnuovo-Mumford regularity is shown cohomologically. That $\tau = s - 1$ if and only if \mathfrak{Z} is on a line is a consequence of ΔH being an O -sequence summing to s . So $\tau = s - 1$ if and only if $\Delta H = (1, 1, \dots, 1)$, which is equivalent to $(H_3)_1 = 2$. P. Maroscia's result (iv) is shown by deforming monomial ideals defining Artin quotients of R' having Hilbert function C (see [Mar, GeM]). The last statement concerning a converse follows from the 1-1 correspondence between saturated ideals of R and subschemes of \mathbb{P}^n . \square

The first difference $\Delta H = (1, c_1, \dots, c_{\sigma-1}, 0, \dots)$ is sometimes termed the h -vector of \mathfrak{Z} (see, for example, [Mig1, §1.4]).

Proposition 1.13. INVERSE SYSTEM OF A PUNCTUAL SCHEME. *The inverse system W is the inverse system of a saturated ideal $I_{\mathfrak{Z}}$, where \mathfrak{Z} a degree- s zero-dimensional scheme of \mathbb{P}^n , regular in degree σ if and only if*

- (i) $\dim_k W_j = s \ \forall j \geq \sigma - 1$, and
- (ii) $\exists N \in \mathbb{N} \mid \forall a, \forall b \geq \max(N, a), \ W_a = R_{b-a} \circ W_b$.

The condition (ii) implies the apparently stronger (1.7). Furthermore, if \mathfrak{Z} is such a degree s scheme regular in degree σ , then for all $b \geq \sigma$,

$$W_b = W_\sigma : R_{b-\sigma} = \{f \in \Gamma_b \mid R_{b-\sigma} \circ f \subset W_\sigma\}. \quad (1.10)$$

Proof. That an inverse system W arising from such a scheme \mathfrak{Z} must satisfy (i),(ii), is immediate from Lemma 1.6, and Theorem 1.12. Suppose conversely that W satisfies (i),(ii). The condition (ii) implies that $I = \text{Ann } (W)$ is a saturated ideal, by Lemma 1.4(ii) applied to (1.6). By (i), its Hilbert polynomial is s , so I defines a zero-dimensional scheme of degree- s having regularity degree no greater than σ . And condition (i) implies that $H(R/I) = (1, \dots, s, s, \dots)$, with the first s occurring before degree $\sigma - 1$. The two imply that R/I is Cohen-Macaulay of dimension 1, and regularity degree no greater than σ (see Theorem 1.12). By Theorem 1.12 (ii) if \mathfrak{Z} is such a scheme, the ideal $I = I_{\mathfrak{Z}}$ is generated by degree σ ; the last equation (1.10) is a translation of this generation fact into the inverse system language, using Lemma 1.4 (i). \square

2 Inverse system of a zero-dimensional scheme.

In Section 2.1 we consider a zero-dimensional scheme $\mathfrak{Z} \subset \mathbb{P}^n$ concentrated at a single point p_0 that is a coordinate point. These are simpler since the local inverse system lies in the ring Γ' . In Section 2.2 we study a scheme \mathfrak{Z} concentrated at an arbitrary point p , for which the local inverse system lies in the completion $\widehat{\Gamma}'$. In Section 2.3 we consider the inverse system for a general zero-dimensional scheme \mathfrak{Z} with finite support. In each case we show how to directly homogenize the local inverse system for \mathfrak{Z} to obtain the global inverse system $L_{\mathfrak{Z}} \subset \Gamma$ of the global defining ideal $I_{\mathfrak{Z}} \subset R$. Recall that we denote by $m_p \subset R$ the homogeneous ideal of the point p ; if

$p = (a_1 : \dots : a_n : 1)$, then $m_p = (a_1 z - x_1, \dots, a_n z - x_n)$. Recall also that the homogeneous ideal $I \subset R$ is concentrated at the point $p \in \mathbb{P}^n$ if and only if there exists an integer $u > 0$ such that

$$m_p \supset I \supset m_p^u. \quad (2.1)$$

When $\text{char } k = 0$ or $\text{char } k > j$ we have ([Ter1], [EmI, Theorem I], [EhR] for $u = 2$)

$$(m_p^u)^\perp \cap \Gamma_j = \Gamma_{u-1} \cdot L_p^{[j+1-u]}. \quad (2.2)$$

Here, the right hand side is interpreted as Γ_j if $u > j$. Thus, the condition (2.1) corresponds to the following condition on the inverse system

$$L_p^{[j]} \subset [I^{-1}]_j \subset \Gamma_{u-1} \cdot L_p^{[j+1-u]}, \quad (2.3)$$

where if $p = (a_1 : \dots : a_n : 1)$ then $L_p = a_1 X_1 + \dots + a_n X_n + X_{n+1}$, and $L_p^{[j]}$ denotes the form $L_p^{[j]} = L_p^j / j! = \sum_{J \mid |J|=j} a^J \cdot X^{[J]}$, proportional to the divided power L_p^j . We have shown

Lemma 2.1. *The following conditions are equivalent:*

- (i) *The homogeneous ideal I of R defines a zero-dimensional scheme concentrated at the point p of \mathbb{P} ;*
- (ii) *There exists an integer u such that $m_p \supset I \supset m_p^u = (a_1 z - x_1, \dots, a_n z - x_n)^u$;*
- (iii) *There exists an integer $\alpha = u - 1$ such that the inverse system I^\perp satisfies*

$$k_{DP}[L_p] \subset I^\perp \subset (m_p^u)^\perp = \Gamma_{\leq \alpha} \cdot k_{DP}[L_p]. \quad (2.4)$$

In particular, if the homogeneous ideal J of R defines a zero-dimensional scheme concentrated at the point $p_0 = (0 : \dots : 0 : 1) \in \mathbb{P}^n$, then $k[Z] \subset J^\perp \subset \Gamma_{\leq a} \cdot k[Z]$, $Z = X_{n+1}$ for some $a \geq 0$.

The following example shows the need for our limitation on the characteristic of k (§1.2).

Example 2.2. Let $n = 1, R = k[x, y], \Gamma = k[X, Y]$. Choose the point $p = (a_1 : 1) \in \mathbb{P}^1$, and $I = m_p^2 = (x - a_1 y)^2$, then we have that $[I^\perp]_2$ satisfies

$$\begin{aligned} (a_1 X + Y)^{[2]} \subset [I^\perp]_2 &\subset \Gamma_1 \cdot L_p = \langle X, Y \rangle \cdot (a_1 X + Y) \\ &= \langle 2a_1 X^{[2]} + XY, a_1 XY + 2Y^{[2]} \rangle, \end{aligned} \quad (2.5)$$

provided $\text{char } k \neq 2$. When $\text{char } k = 2$ and $a_1 = 0$ the space on the right is just $\langle XY \rangle$, so is one-dimensional, and is not all of $(m_p^2)^\perp$, which also includes $L_p^{[2]} = a_1^2 X^{[2]} + a_1 XY + Y^{[2]}$. Thus, equation (2.2) and the equality on the right of Lemma 2.1 (2.4) do not extend to characteristic $p \neq 0$, when p is less than or equal to the degree j (here $j = 2$) of the forms being considered.

Recall that the *socle degree* α of a local Artin algebra A of maximal ideal m is the highest integer such that $m^\alpha A \neq 0$, but $m^{\alpha+1} A = 0$, and that the point $p_0 = (0 : \dots : 0 : 1)$. For a zero-dimensional scheme \mathfrak{Z} , we now define $\alpha(\mathfrak{Z})$ to be the maximum local socle degree of \mathfrak{Z} .

Definition 2.3. (i) If \mathfrak{Z} is a zero-dimensional scheme concentrated at p_0 , we let $\alpha(\mathfrak{Z})$ denote the highest socle degree of (R'/J) , where $J \subset R'$ defines \mathfrak{Z} . Equivalently, $\alpha(\mathfrak{Z})$ is the highest degree of an element of $J^{-1} \in \Gamma'$. If \mathfrak{Z} is concentrated at a point p , then $\alpha(\mathfrak{Z})$ is defined similarly using the local ring at p (see Section 2.2).

(ii) More generally, if a zero-dimensional scheme \mathfrak{Z} has decomposition $\mathfrak{Z} = \mathfrak{Z}(1) \cup \dots \cup \mathfrak{Z}(k)$ as the union of irreducible components $\mathfrak{Z}(1), \dots, \mathfrak{Z}(k)$, each concentrated at (distinct) points $p(1), \dots, p(k)$, then $\alpha(\mathfrak{Z}) = \max\{\alpha(\mathfrak{Z}(1)), \dots, \alpha(\mathfrak{Z}(k))\}$ of the local socle degrees.

2.1 Schemes concentrated at a coordinate point.

We will fix the coordinate point as $p = p_0 = (0 : \dots : 0 : 1)$; we denote x_{n+1}, X_{n+1} by z, Z , respectively. We let $R' = \mathbb{k}[y_1, \dots, y_n]$ be the coordinate ring of affine space \mathbb{A}^n , the locus on \mathbb{P}^n where $x_{n+1} \neq 0$ and we let $\Gamma' = \mathbb{k}_{DP}[Y_1, \dots, Y_n]$ be the divided power ring. Let $\mathcal{I}_p \subset \mathcal{O}_p$ be an ideal defining a zero-dimensional scheme \mathfrak{Z} concentrated at p . Then $\mathcal{I}_p \supset m_p^{\alpha(\mathfrak{Z})+1}$ and each element of \mathcal{I}_p may be written mod $m_p^{\alpha(\mathfrak{Z})+1}$ as a polynomial h in $R' = \mathbb{k}[y_1, \dots, y_n]$ of some degree t no greater than $\alpha(\mathfrak{Z})$. The homogenization of h to degree u is

$$\text{Homog}(h, z, u) = z^u \cdot h(x_1/z, \dots, x_n/z), \quad (2.6)$$

for $u \geq t$, and 0 otherwise. The homogenization $I_{\mathfrak{Z}}$ of \mathcal{I}_p is spanned by $m_p^{\alpha(\mathfrak{Z})+1}$, and by all homogenizations of such elements $h \in \mathcal{I}_p$:

$$I_{\mathfrak{Z}} = (\text{Homog}(h, z, u) \mid u \in \mathbb{Z}^+, h \in \mathcal{I}_p \text{ degree } h \leq \alpha(\mathfrak{Z})) + m_p^{\alpha(\mathfrak{Z})+1} \quad (2.7)$$

Recall that the inverse system $L_{\mathfrak{Z}}$ in Γ of $I_{\mathfrak{Z}}$ consists of all elements of Γ , annihilated by $I_{\mathfrak{Z}}$. Given a point $p = (a_1 : \dots : a_n : 1)$ of \mathbb{P}^n , we let $L_p = a_1 X_1 + \dots + a_n X_n + Z \in \Gamma$.

Definition 2.4. *Homogenization of an inverse system at a point.*

- (i) Let $F \in \Gamma[1/Z, 1/Z^{[2]}, \dots]$. We denote by $F \cdot_{rp} Z^{[u]}$ the result of raising the Z -degree of the Z -factor in each term by u , without changing the coefficients that appear. For example, if $F = X_1 X_2 / Z^{[2]} + X_2^{[4]} / Z^{[4]} \in \mathbb{k}_{DP}[Y_1, Y_2]$, then $F \cdot_{rp} Z^{[4]} = X_1 X_2 Z^{[2]} + X_2^{[4]}$. We may also write $Z^{[u]} \cdot_{rp} F$ for $F \cdot_{rp} Z^{[u]}$. If $w \in \Gamma$ has the form $w = \sum w_i \cdot L_p^{[k-i]}$, $w_i \in \Gamma'$, then we denote by $w \cdot_{rp} L_p^{[u]}$ the product

$$w \cdot_{rp} L_p^{[u]} = \sum w_i \cdot L_p^{[k+u-i]}. \quad (2.8)$$

- (ii) Let $f \in \Gamma' = \mathbb{k}_{DP}[Y_1, \dots, Y_n]$ satisfy $f = \oplus f_i, f_i \in \Gamma'_i$, and let $L_p = a_1 X_1 + \dots + a_n X_n + Z$. Then for any integer $u \geq 0$ we define the inverse system homogenization

$$\text{Homog}(f, L_p, u) = \sum_{0 \leq i \leq u} f_i(X_1, \dots, X_n) \cdot L_p^{[u-i]}. \quad (2.9)$$

For example, if $f = Y_1 Y_2 + Y_2^{[4]}$, then $f(X_1/Z, X_2/Z) = F$ above, and $\text{Homog}(f, Z, 4) = X_1 X_2 Z^{[2]} + X_2^{[4]}$, while $\text{Homog}(f, Z, 3) = X_1 X_2 Z$.

- (iii) Let $L' \subset \Gamma'$ be an inverse system (so L' is an R' -submodule of Γ'), and suppose p fixed. Then we define

$$L'[u] = \langle \{\text{Homog}(f, L_p, u) \mid f \in L'\} \rangle$$

and we define the homogenization of the inverse system L' ,

$$\text{Homog}(L', L_p) = \bigoplus_{u \geq 0} L'[u] = \langle \text{Homog}(f, L_p, u) \mid f \in L', u \geq 0 \rangle. \quad (2.10)$$

If we leave out the homogenizing form or do not specify p , then we assume $L_p = Z, p = p_0$.

Note that this definition allows $\text{Homog}(f, Z, u)$ to be nonzero even if u is smaller than the degree of f ; this is natural here, since the global inverse system is closed under the contraction action of R . Thus, for example

$$z \circ (X_1 X_2 Z^{[2]} + X_2^{[4]}) = X_1 X_2 Z.$$

We of course wish to show that if $L' \subset \Gamma'$ is the inverse system of $\mathcal{I}_p \subset \mathcal{O}_p$, then $L = \text{Homog}(L', Z) \subset \Gamma$ is the inverse system of $I_{\mathfrak{Z}}$ (Lemma 2.7). We also wish to show how to obtain from L' the key *generators* of L — which is infinitely generated. To this end, we need a basic result.

Lemma 2.5. HOMOGENIZATION AND DUALITY. Suppose that $h' \in R'$ has degree a , that $f' \in \Gamma'$, that $i \geq a$, and that $w \in \mathbb{Z}$. Let $h = h'[i] = \text{Homog}(h', z, i)$ and $f = f'[i+w] = \text{Homog}(f', Z, i+w)$. Then

$$h \circ f = h'[i] \circ f'[i+w] = (h' \circ f')[w]. \quad (2.11)$$

In particular,

$$h' \circ f' = 0 \Rightarrow (h' \circ f')_{\leq w} = 0 \Leftrightarrow (h' \circ f')[w] = 0 \Leftrightarrow h'[i] \circ f'[i+w] = 0; \quad (2.12)$$

and if f' has degree b , then

$$h' \circ f' = 0 \Leftrightarrow (h' \circ f')_{\leq b} = 0 \Leftrightarrow (h' \circ f')[b] = 0 \Leftrightarrow h'[i] \circ f'[i+b] = 0. \quad (2.13)$$

Proof. Let $h' = \sum_{u=0}^a h_u$ and $f' = \sum_{v=0}^b f_v$. Then $h = h'[i] = \sum_{u=0}^a h_u z^{i-u}$ and $f'[i+w] = \sum_{v=0}^{\min\{b, i+w\}} f_v Z^{[i+w-v]}$. Now, we have formally (below, $Z^{[c]} = 0$ if $c < 0$),

$$\begin{aligned} h'[i] \circ f'[i+w] &= \sum_{u=0}^a \left(\sum_{v=u}^{\min\{b, i+w\}} h_u(x_1, \dots, x_n) \circ f_v(X_1, \dots, X_n) \cdot Z^{[w+u-v]} \right) \\ &= \sum_{u=0}^a \left(\sum_{v=u}^b h_u \circ f_v \cdot Z^{[w+u-v]} \right) = h' \circ f'[w]. \end{aligned}$$

The second equation is immediate from the first, since homogenization to degree w in Γ' annihilates terms in $h \circ f$ having degree greater than w . The third is immediate from the second. \square

Lemma 2.6. Suppose $h' \in R'$ has degree no greater than a , and $f' \in \Gamma'_{\leq b}$, and let $h = \text{Homog}(h', z, a) \in R$, $f = \text{Homog}(f', Z, b) \in \Gamma$. Then

$$h' \circ f' = 0 \Leftrightarrow h \circ (f \cdot_{rp} Z^{[a]}) = 0. \quad (2.14)$$

Proof. In (2.13), take $i = a$, and note that $\text{Homog}(f', Z, a+b) = \text{Homog}(f', Z, b) \cdot_{rp} Z^{[a]}$. \square

Lemma 2.7. LOCAL TO GLOBAL INVERSE SYSTEMS. Suppose that $p = p_0 = (0 : \dots : 0 : 1)$ in \mathbb{P}^n , and that $L' \subset \Gamma'$ is the inverse system of $\mathcal{I}_p \subset \mathcal{O}_p$, where \mathcal{I}_p defines a degree- s zero-dimensional scheme \mathfrak{Z} concentrated at p . Then $\text{Homog}(L', Z) \subset \Gamma$ is the inverse system $L_{\mathfrak{Z}}$ of $I_{\mathfrak{Z}} \subset R$.

Proof. Let $S = \text{Homog}(L', Z)$. It is immediate from (2.12) in Lemma 2.5 that $I_{\mathfrak{Z}} \circ S = 0$, so $L_{\mathfrak{Z}} \supset S$. Also, note that S is an R -module: $R \circ S \subset S$. To show this, it suffices to check that if $f \in S_u$, $h \in R_1$, then $h \circ f \in S_{u-1}$. Let $f = \text{Homog}(f', Z, u)$, $f' \in L'$. Note that $z \circ f = \text{Homog}(f', Z, u-1)$, so is in S . Also, if $1 \leq i \leq n$ then considering each term, it is easy to see that $x_i \circ f = \text{Homog}(y_i \circ f', Z, u-1)$, so is in S . This shows $R_1 \circ S \subset S$, and by induction that S is an R -module.

For $i \geq \tau(\mathfrak{Z})$, $\dim_{\mathbf{k}}(L_{\mathfrak{Z}})_i = s$. For $i \geq \alpha(\mathfrak{Z})$, the socle degree, $\dim_{\mathbf{k}} S_i = s$, since the homomorphism $f \in L' \rightarrow f[i]$ is an isomorphism of $\Gamma'_{\leq i}$ into Γ_i , and $\dim_{\mathbf{k}} L' = s$. Since $S \subset L_{\mathfrak{Z}}$, we have $S_i = (L_{\mathfrak{Z}})_i$ for $i \geq \max\{\alpha(\mathfrak{Z}), \tau(\mathfrak{Z})\}$. Since $\bar{I}_{\mathfrak{Z}}$ is saturated, by Lemma 1.6 we have that there is an integer N such that $(L_{\mathfrak{Z}})_u = R_{i-u} \circ (L_{\mathfrak{Z}})_i$ for all $i \geq N$ and $u \leq i$. We conclude that $L_{\mathfrak{Z}} \subset S$, completing the proof of the Lemma. \square

The following result is a consequence of Lemma 2.7. We give a direct proof.

Lemma 2.8. When $j \geq \alpha(\mathfrak{Z})$, $(L_{\mathfrak{Z}})_{\geq j}$ is closed under the raised power action of Z : $f \rightarrow f \cdot_{rp} Z^{[u]}$.

Proof. Let $f \in (L_3)_j$, set $f_1 = f \cdot_{rp} Z$, $I = I_3$, and suppose by way of contradiction that $h \in I_{j+1}$ satisfies $h \circ f_1 \neq 0$. Then $h = zh_1 + h'$, $h_1 \in R_j$, $h' \in R'_{j+1}$. Since $j \geq \alpha(\mathfrak{Z})$, $h' \in J_{j+1}$, where J is the ideal defining $\mathfrak{Z} \subset \mathbb{A}^n$; hence $zh_1 \in I_{j+1}$, implying $h_1 \in I_j$, since the homogenizing variable is a non-zero divisor of R/I_3 . But we have $h' \circ f_1 = 0$ (as each term of f_1 has a Z -factor), hence $zh_1 \circ f_1 = (h \circ f_1 - h' \circ f_1) \neq 0$. Then $h_1 \circ f = zh_1 \circ f_1 \neq 0$, a contradiction since $h_1 \in I_j$. \square

The assumption $j \geq \alpha(\mathfrak{Z})$ in the above Lemma is necessary (see Example 2.17). We now state a key result concerning the generation of the homogenized inverse system.

Lemma 2.9. GENERATORS FOR THE GLOBAL INVERSE SYSTEM. *Suppose that $V' \subset \Gamma'_{\leq \alpha}$ generates the inverse system L' of \mathcal{I}_p , and denote by I_3 the homogenization of \mathcal{I}_p , and by V the subspace $\text{Homog}(V', Z, \alpha)$ of Γ_α . Then the inverse system $L_3 = I_3^{-1} \subset \Gamma$ satisfies*

$$\begin{aligned} (L_3)_j &= \text{Homog}(L'_{\leq \alpha}, Z, j) \\ &= R_\alpha \circ (V \cdot_{rp} Z^{[j]}). \end{aligned} \quad (2.15)$$

Proof. Since $L' = L'_{\leq \alpha}$, the first equality follows from Lemma 2.7. That V' generates L' is equivalent to $L' = R'_{\leq \alpha} \circ V'$. If $h' \in R'_{\leq \alpha}$ and $v' \in V'$, let $v = v'[\alpha]$; then by Lemma 2.5 $h'[\alpha] \circ (v \cdot_{rp} Z^{[j]}) = h'[\alpha] \circ v'[j + \alpha] = (h' \circ v')[j] \in \text{Homog}(L', Z, j)$. This shows $\text{Homog}(L', Z, j) \subset R_\alpha \circ (V \cdot_{rp} Z^{[j]})$. Lemmas 2.8 and 2.7 show that $V \cdot_{rp} Z^{[j]} \subset \text{Homog}(L', Z, j + \alpha) = (L_3)_{j+\alpha}$, implying the opposite inclusion. This completes the proof of (2.15). \square

Example 2.10. The above Lemma 2.9 can be used to calculate the homogenization of an ideal, given generators of the local inverse system. Begin with the local ideal $I' \subset R' = \mathbb{k}[y_1, y_2]$, $I' = \text{Ann}(f')$, $f' = Y_1^{[8]} + Y_2^{[8]} + Y_1^{[3]}Y_2^{[3]} + (Y_1 + Y_2)^{[6]}$. Then

$$I' = (3y_1^6 - 4y_1y_2^5 + y_2^6 + y_1^2y_2^2 - 2y_1y_2^3, y_1^6 - y_2^6 + y_1^3y_2 - y_1y_2^3, m_{p_0}^9),$$

of local Hilbert function $H' = H(R'/I') = (1, 2, 3, 4, 3, 2, 2, 1)$, and I' defines a degree-20 zero-dimensional scheme $\mathfrak{Z} = \text{Spec}(R'/I')$ concentrated at $p_0 = (0 : 0 : 1) \in \mathbb{P}^2$, with $\alpha(\mathfrak{Z}) = 8$. The homogenized ideal $I = I_3 \subset R$ has more than two generators, and is tricky to find directly — we may homogenize a standard basis, using a computer algebra program. However, by homogenizing f' , forming $f = \text{Homog}(f', Z, 8) = X_1^{[8]} + X_2^{[8]} + X_1^{[3]}X_2^{[3]}Z^{[2]} + (Y_1 + Y_2)^{[6]}Z^{[2]}$, we may calculate $W_8 = R_8 \circ (f \cdot_{rp} Z^{[8]})$, and we can find $J = \text{Ann } W_8$. In the MACAULAY algebra program [BGS] we found the contraction of R_8 with f , then used the script “<lfrom_dual” to find J . The homogenized ideal $I = J_{\leq 8} + m_p^9$, where $m_p = (x_1, x_2)$. In this case $J_{\leq 8}$ already generates I , since $\Delta(H(R/J_{\leq 8}))$ has the correct degree 20. We found $I = I_3$ satisfies

$$\begin{aligned} I = (x_1^3x_2^2 + x_1^2x_2^3 - 3x_1x_2^4, & x_1^4x_2 - x_1x_2^4, x_1x_2^5 - x_2^6 + (3/4)x_1^3x_2z^2 - (1/4)x_1^2x_2^2z^2 - (1/4)x_1x_2^3z^2, \\ & x_1^6 - x_2^6 + x_1^3x_2z^2 - x_1x_2^3z^2), \end{aligned}$$

of Hilbert function $H_3 = H(R/I)$ satisfying $\Delta H_3 = (1, 2, 3, 4, 5, 4, 1)$, and $\sigma(\mathfrak{Z}) = 7$.

The following Proposition extends some of the above results. Recall that we use the notation z for x_{n+1} and Z for X_{n+1} and we denote by m_z or m_{z^a} multiplication by x_{n+1} or by x_{n+1}^a in R or in $A = R/I$. We denote by $L = \text{Homog}(J^{-1}, Z) \subset \Gamma$, the homogenization of the inverse system $J^{-1} \subset \Gamma' = \mathbb{k}_{DP}[Y_1, \dots, Y_n]$ and we let $L_i[j] = \text{Homog}(L_{i\{X_{n+1}=1\}}, X_{n+1}, j)$. This is just $L_i[j] = z^{i-j} \circ L_i \in \Gamma_j$ if $j \leq i$, and $L_i[j] = Z^{[j-i]} \cdot_{rp} L_i$ if $j \geq i$ and is obtained in any case by changing each $X_{n+1}^{[u]}$ factor appearing in a monomial term of an element $F \in L_i$ to $X_{n+1}^{[u+j-i]}$, forming an element $F[j] \in \Gamma_j$. Note that if $j > i$, and $F \in L_i$, then $F[j]$ is not necessarily in L_j (see Remark 2.12 and Example 2.15 below). Recall that M' is the maximal ideal of $R' = \mathbb{k}[y_1, \dots, y_n]$ at the origin, and $m_p = (x_1, \dots, x_n) \subset R$ is the homogeneous maximal ideal of R at the corresponding point $p \in \mathbb{P}^n$.

Proposition 2.11. HOMOGENIZATION FOR SCHEMES WITH SUPPORT p_0 . Let $J \subset R'$ define a zero-dimensional scheme $\mathfrak{Z} = \text{Spec}(R'/J)$ concentrated at the origin and let $\alpha = \alpha(\mathfrak{Z})$ be the socle degree of R'/J (Definition 1.7). Let $L' = J^{-1} \subset \Gamma'$ be the affine inverse system of \mathfrak{Z} , and set $I = \text{Homog}(J, z) \subset R$, $L = \text{Homog}(L', Z) = \bigoplus_i L'[i]$. Then we have

- (i) $I = I_{\mathfrak{Z}}$, and is a saturated ideal primary to the maximal ideal $m_p, p = (0 : \dots : 0 : 1) \in \mathbb{P}^n$, and it satisfies $m_p \supset I \supset m_p^{\alpha+1}$. In particular, $I_a = (I_b : R_{b-a})$ for $a \leq b$, and $I_b = R_{b-a}I_a$ for $b \geq a \geq \alpha(\mathfrak{Z})$. Furthermore, if $a \leq b$, then $I_a = (I_b : z^{b-a})$, and if $b \geq \alpha$ we have $I_b = z^{b-\alpha} \cdot I_{\alpha} + (m_p^{\alpha+1})_b$.
- (ii) $L = L_{\mathfrak{Z}}$ and satisfies $L_a = z^{b-a} \circ L_b$ for $a \leq b$, and $\mathbf{k}_{DP}[Z] \subset L \subset \Gamma_{\leq \alpha} \cdot \mathbf{k}_{DP}[Z]$. Furthermore, for $\alpha \leq a \leq b$ the map $F \rightarrow F[b]$ taking L_a to L_b , and the map $z^{b-a} \circ : L_b \rightarrow L_a$ are inverse isomorphisms. Also, L satisfies $L_b = L_a[b]$ for any pair (a, b) satisfying $a \geq \alpha$.
- (iii) A satisfies $m_{z^{b-a}} : A_a \rightarrow A_b$ is injective for $a \leq b$, and furthermore $m_{z^{b-a}}$ defines an isomorphism $A_a \cong A_b$ for $\alpha \leq a \leq b$. In particular, for $k > 0$, $m_{z^k} : A_{\alpha-k} \rightarrow A_{\alpha}$ is an injection, and $m_{z^k} : A_{\alpha} \rightarrow A_{\alpha+k}$ is an isomorphism onto.
- (iv) Let $\dim_{\mathbf{k}} R'/J = s$. The subscheme $\mathfrak{Z} = \text{Proj}(A)$ of \mathbb{P}^n has degree s , and $J \supset M'^s$. Furthermore, the regularity $\sigma(\mathfrak{Z})$ satisfies $\sigma(\mathfrak{Z}) = \tau(\mathfrak{Z}) + 1 \leq \alpha + 1 \leq s$.
- (v) Let $\mathfrak{L} \subset \Gamma$ be an inverse system satisfying $\mathbf{k}_{DP}[Z] \subset \mathfrak{L} \subset \Gamma_{\leq \alpha} \cdot \mathbf{k}_{DP}[Z]$, and let $\mathfrak{J} = \text{Ann}(\mathfrak{L})$. Then, letting $\mathfrak{J} = (\mathfrak{J})_{(z=1)} \subset R'$, and $\mathfrak{L}' = (\mathfrak{L}_b)_{Z=1, b \geq \alpha}$ we have that $\mathfrak{L}' = (\mathfrak{J})^{-1}$, \mathfrak{J} defines a scheme \mathfrak{Z} concentrated at the origin with $\mathfrak{J} = I_{\mathfrak{Z}}, \mathfrak{L} = L_{\mathfrak{Z}}, \alpha(\mathfrak{Z}) \leq \alpha$, and $\mathfrak{L} = \text{Homog}(\mathfrak{L}', Z)$.

Proof. That $I = I_{\mathfrak{Z}}$ and is saturated is well-known, since the primary decomposition of an ideal carries over to its homogenization (see §VII.5 Theorem 17 of [ZarS]). The next statements of (i) are standard, since z is a non-zero divisor in $R/I_{\mathfrak{Z}}$. That $J = (J_{\leq \alpha}) + (M')^{\alpha+1}$ implies the last statement of (i).

That $L = L_{\mathfrak{Z}}$, and $\mathbf{k}_{DP}[Z] \subset L \subset \Gamma_{\leq \alpha} \cdot \mathbf{k}_{DP}[Z]$ in (ii) follow from Lemma 2.9 and (i). That $L_a = z^{b-a} \circ L_b$ follows from z being a non-zero divisor in $R/I_{\mathfrak{Z}}$, and Lemma 1.6. Lemma 2.8 and an easy verification implies that the two maps given are inverse isomorphisms when $a, b \geq \alpha$. For any $f \in J^{-1}$, $\deg f \leq \alpha$, so if $a \geq \alpha$ we have $f[b] = (f[a])[b]$ and this implies the last statement of (ii). The statements of (iii) follow from and are weaker than those of (i) or (ii); they are about the quotient algebra A , rather than the ideal I or inverse system L . Now (ii) and (iii) imply the key inequality $\tau(\mathfrak{Z}) \leq \alpha(\mathfrak{Z})$ of (iv) since $\tau(\mathfrak{Z}) = \min\{i \mid \dim_{\mathbf{k}}(R/I_{\mathfrak{Z}})_i = s\}$. That $J \supset M'^s$ is well known; any monomial of M'^s has a length- $(s+1)$ chain of monomials that divide it, so some linear combination of elements of the chain must be in J — since the degree of \mathfrak{Z} is only s — implying the monomial itself is in J .

Note that the main condition of (v) is that of (2.3) with $L_p = Z$; this is the condition for \mathfrak{J} to define a zero-dimensional scheme \mathfrak{Z} concentrated at p_0 , so (v) would follow from the standard fact, $J = (I_{\mathfrak{Z}})_{z=1}$ defines the portion of \mathfrak{Z} in $\mathbb{A}^n : z = 1$, and (ii), provided we show that $\mathfrak{L}' = J^{-1}$. Directly, we have $1 \subset \mathfrak{L}' \subset \Gamma'_{\leq \alpha}$; thus, identifying x and y variables (since we have taken $z = 1$) and letting $\mathfrak{J}' = \text{Ann}(\mathfrak{L}') \subset R'$, we have $(x_1, \dots, x_n) \supset \mathfrak{J}' \supset (x_1, \dots, x_n)^{\alpha+1}$. Also, we have $\dim_{\mathbf{k}} \mathfrak{L}' = \dim_{\mathbf{k}} \mathfrak{L}_b = H(R/I_{\mathfrak{Z}})_b = s$, with $s = \deg(\mathfrak{Z})$, as there is no kernel in dehomogenizing from a vector subspace of Γ_b . Clearly \mathfrak{L}' is independent of the choice of $b \geq \alpha$ by (ii). Taking $b = 2\alpha$ and using equation (2.13) of Lemma 2.5 we can see that $\mathfrak{J} \subset \mathfrak{J}'(Y)$, but we have $\dim_{\mathbf{k}}(\mathbf{k}[x_1, \dots, x_n]/\mathfrak{J}') = \dim_{\mathbf{k}}(R/\mathfrak{J}) = s$, implying $\mathfrak{J} = \mathfrak{J}'$. It likewise follows from the equality of dimensions that $\mathfrak{L}' = \mathfrak{J}^{-1}$. This completes the proof of (v), and of Proposition 2.11. \square

Remark 2.12. Homogenized component L_i is not determined by L'_i . We note here a perhaps surprising property of the homogenized inverse system $L = \text{Homog}(J^{-1}, Z)$, where $J^{-1} \subset \Gamma'$ is the inverse system of an ideal $J \subset R'$ defining a zero-dimensional scheme \mathfrak{Z} . Namely, $F \in L_i, i < \alpha(\mathfrak{Z})$,

does not imply that there is a (possibly nonhomogenous) element $f \in J^{-1}$ of degree i such that $F = f[i]$. There are also elements of L_i arising from homogenizing to degree i those elements of J^{-1} having higher degree. See Example 2.15 below, where $X_1^{[2]} \in L_2$, $X_1^{[2]} = z \circ (X_1^{[2]}Z - X_1X^{[2]})$, but is not a homogenization of an element of $J_{\leq 2}^{-1}$. Likewise, as mentioned earlier, $X_1^{[2]} \in L_2$ does *not* imply $\text{Homog}(X_1^{[2]}, Z, 3) = X_1^{[2]}Z \in L_3$; rather the corresponding element of L_3 is $X_1^{[2]}Z - X_1X_2^{[2]}$. However, if $i \geq \alpha(\mathfrak{Z})$, then $F \in L_i$ and if $j \geq i$, $F[j] \in L_j$ by Proposition 2.11(ii). For similar reasons, the condition $b \geq \alpha$ in Proposition 2.11(v) cannot be removed, and we may have $((L_3)_a)_{Z=1} \not\subset ((L_3)_b)_{Z=1}$ when $a < b$. (See Example 2.15 below).

Note that $\sigma(\mathfrak{Z})$ may be rather less than $\alpha + 1$, the upper bound of (iv), and is almost always less than $\alpha + 1$ when the defining ideal of \mathfrak{Z} in R' is non-homogeneous. (See Examples 2.10, 2.13, 2.17).

If we write $\alpha(\mathfrak{Z}) = \sigma(\mathfrak{Z}) + k(\mathfrak{Z})$, it is not clear how to bound above $k(\mathfrak{Z})$. The examples where \mathfrak{Z} is defined locally by a general enough compressed Gorenstein ideal of \mathcal{O}_p , in the sequel article [ChoI2] show that there is no constant upper bound. On the other hand, these examples satisfy $k(\mathfrak{Z}) \leq \sigma(\mathfrak{Z})$, suggesting that the latter bound might be valid for \mathfrak{Z} supported at a single point.

For any zero-dimensional scheme \mathfrak{Z} , Proposition 1.13 shows that the inverse system L of $I_{\mathfrak{Z}}$ is determined by L_{σ} ; thus $L_i = (L_{\sigma} : R_{i-\sigma})$ if $i \geq \sigma$, and $L_i = R_{\sigma-i} \circ L_{\sigma}$ if $i \leq \sigma$. However, when both $\alpha, i > \sigma$, L_i may not be obtained by simply raising the Z -power of elements of L_{σ} even when \mathfrak{Z} is concentrated at p_0 (see Example 2.17).

Below we set $Z^{[u]} = 0$ if $u < 0$.

Example 2.13. L_{τ} may not determine L . Let $R = \mathbb{k}[x_1, x_2, x_3]$, $p = p_0 = (0 : 0 : 1)$, $\mathcal{I}_p = (y_1y_2, y_1^2 - y_2^3)$, $f' = (Y_1^{[2]} + Y_2^{[3]})$; then $\mathcal{I}_p \supset (y_1y_2, y_1^3, y_2^4)$ and $H(R_p/\mathcal{I}_p) = (1, 2, 1, 1)$, and $\alpha(\mathfrak{Z}) = 3$. The homogenization $I_{\mathfrak{Z}} = (x_1x_2, x_1^2z - x_2^3, x_1^3, x_2^4)$, and $H(R/I_{\mathfrak{Z}}) = (1, 3, 5, 5, \dots)$, so $\tau(\mathfrak{Z}) = 2$, $\sigma(\mathfrak{Z}) = 3$. The inverse system $L = I_{\mathfrak{Z}}^{-1}$ satisfies, by Lemma 2.9

$$\begin{aligned} L_j &= \langle X_1^{[2]}Z^{[j-2]} + X_2^{[3]}Z^{[j-3]}, X_2^2Z^{[j-2]}, X_2Z^{[j-1]}, X_1Z^{[j-1]}, Z^{[j]} \rangle \\ &= R_3 \circ (\text{Homog}(f', Z, j+3)) = R_3 \circ (X_1^{[2]}Z^{[j+1]} + X_2^{[3]}Z^{[j]}) \\ &= \text{Homog}(V', Z, j), \text{ where } V' = R' \circ f' = \langle f', Y_2^{[2]}Y_2, Y_1, 1 \rangle. \end{aligned}$$

Note that $L_{\sigma} = L_3$ determines L , but the space $L_{\tau} = L_2$ does not. This corresponds to $(I_{\mathfrak{Z}})_{\sigma}$ determining $I_{\mathfrak{Z}}$ (see Theorem 1.12(ii)). Also, since $\Delta H(R/I_{\mathfrak{Z}}) = (1, 2, 2, 0)$, which is not symmetric, \mathfrak{Z} is not arithmetically Gorenstein; however, \mathfrak{Z} is locally Gorenstein and has a single point of support.

Example 2.14. More generally, with R, p as above, let $f' = (Y_1^{[2]} + Y_2^{[j]})$, $j \geq 3$; then $\mathcal{I}_p = (y_1y_2, y_1^2 - y_2^j)$ and $H(R_p/\mathcal{I}_p) = (1, 2, 1, \dots, 1_j)$, with $j-1$ ones at the end, determining a zero-dimensional scheme \mathfrak{Z} at p of degree $j+2$, for which $\alpha(\mathfrak{Z}) = j$. The homogenization $I_{\mathfrak{Z}} = (x_1x_2, x_1^2z^{j-2} - x_2^j, x_1^3)$, so $H = (1, 3, 5, 6, \dots, j+2, j+2, \dots)$, and $\Delta H(R/I_{\mathfrak{Z}}) = (1, 2, 2, 1, \dots, 1, 0)$, with $j-3$ ones at the end, so $\tau(\mathfrak{Z}) = j-1$, $\sigma(\mathfrak{Z}) = j$. The inverse system $L = I^{-1}$ is determined by

$$L_{\sigma} = \langle X_1^{[2]} \cdot Z^{[j-2]} + X_2^{[j]}, X_2^{[j-1]}Z, \dots, X_2Z^{[j-1]}, X_1Z^{[j-1]}, Z^{[j]} \rangle,$$

in the same sense as Example 2.13, but is not so determined by L_{τ} .

When $j = 3$ or $j \geq 5$ then \mathfrak{Z} is not arithmetically Gorenstein, since $\Delta H(R/I_{\mathfrak{Z}})$ is not symmetric. When $j = 4$ then $H_{\mathfrak{Z}} = (1, 3, 5, 6, 6, \dots)$, $\Delta H = (1, 2, 2, 1)$, and \mathfrak{Z} is arithmetically Gorenstein if and only if it satisfies the Cayley-Bacharach property that $H(R/I_{\mathfrak{Z}'})_{\tau-1} = H(R/I_{\mathfrak{Z}})_{\tau-1}$ for any subscheme $\mathfrak{Z}' \subset \mathfrak{Z}$ of degree $s-1$ (see [Kr2], [Mig1, Theorem 4.1.10]). Here $\tau(\mathfrak{Z}) = 3$, $\sigma(\mathfrak{Z}) = 4$, and the local inverse system is $L' = \langle 1, Y_1, Y_2, Y_2^{[2]}, Y_2^{[3]}, Y_1^{[2]} + Y_2^{[4]} \rangle$. The only R' -closed system L'' of codimension one is $L'' = \langle 1, Y_1, Y_2, Y_2^{[2]}, Y_2^{[3]} \rangle$. Since L'' , and thus $J'' = \text{Ann}(L'') \subset R''$ defining

the subscheme \mathfrak{Z}'' is graded, Proposition 2.18 implies that $H(R/I_{\mathfrak{Z}''}) = (1, 3, 4, 5, 5, \dots)$, the sum function of $H(R'/J'') = (1, 2, 1, 1)$. Thus \mathfrak{Z} does not satisfy the Cayley-Bacharach condition, and is not arithmetically Gorenstein.

Example 2.15. *Component L_2 not the homogenization of L'_2 .* If $r=3$, $R = k[x_1, x_2, x_3]$, $\Gamma' = k[Y_1, Y_2]$, $f = Y_1^{[2]} - Y_1 Y_2^{[2]} \in \Gamma'$, then $I' = (y_1^2 + y_1 y_2^2, y_2^3)$, of Hilbert function $H(R'/I') = (1, 2, 2, 1)$, and $\alpha(\mathfrak{Z}) = 3$. The related homogeneous ideal I in R determining the degree-6 scheme \mathfrak{Z} concentrated at $p_0 = (0 : 0 : 1)$ in \mathbb{P}^2 is

$$I = (x_1^2 z + x_1 x_2^2, x_2^3, x_1^3, x_1^2 x_2),$$

of Hilbert function $H_3 = (1, 3, 6, 6, \dots)$, so $\tau(\mathfrak{Z}) = 2, \sigma(\mathfrak{Z}) = 3$. Here the homogenization of f to degree α is $G = f[3] = X_1^{[2]} Z - X_1 X_2^{[2]}$. By Lemma 2.9, letting $L = L_3 = I_3^{-1}$, we have that L is simply determined by the actions of the pair $(z, Z) = (x_3, X_3)$ on L_α , which satisfies

$$\begin{aligned} L_3 &= R_3 \circ F[6] = R_3 \circ (X_1^{[2]} Z^{[4]} - X_1 X_2^{[2]} Z^{[3]}) \\ &= \langle G, Z^{[3]}, X_1 Z^{[2]}, X_1 X_2 Z, X_2^{[2]} Z, X_2 Z^{[2]} \rangle. \end{aligned}$$

Likewise, $L_2 = R_1 \circ L_3 = \langle X_1^{[2]}, X_1 Z, X_1 X_2, X_2^{[2]}, X_2 Z, Z^{[2]} \rangle \subset \Gamma_2$. Note that L_2 contains $X_1^{[2]}$, which is the partial of $G = f[3]$ with respect to z , but is not the homogenization of an element of $L'_{\leq 2} = I'^{-1}_{\leq 2}$, as $L' = \langle f, Y_2^{[2]}, Y_1 Y_2, Y_2, Y_1, 1 \rangle$.

Example 2.16. When R', p are as above, and $I' = (y_1^2, y_2^3)$, $f' = Y_1 Y_2^{[2]}$, the local Hilbert function is $H(R'/I') = (1, 2, 2, 1)$, $\alpha(\mathfrak{Z}) = 3$, then $I = I_3 = (x_1^2, x_2^3)$, $H(R/I_3) = (1, 3, 5, 6, 6, \dots)$, so $\sigma(\mathfrak{Z}) = 4, \tau(\mathfrak{Z}) = 3 = \alpha(\mathfrak{Z})$, and $L = I^{-1}$ is determined by

$$L_\tau = \langle X_1 X_2^{[2]}, Z X_2^{[2]}, Z^{[2]} X_2, Z X_1 X_2, Z^{[2]} X_1, Z^{[3]} \rangle,$$

in the stronger sense that $L_j = R_\tau \circ (L_\tau \cdot_{rp} Z^j)$ if $j \geq \tau$, and $L_j = R_{\tau-j} \circ L_\tau$ when $j \leq \tau$. This example and Example 2.13 above illustrate that L must be determined by L_σ , but L is also determined by L_τ if I is generated in degrees less or equal to τ . The next example shows that this determination by L_τ (or by L_σ) is usually in a *weaker* sense than here.

Example 2.17. *How does L_σ determine L ?* We choose a curvilinear ideal (one with $\mathcal{R}_p/\mathcal{I}_p \cong k[y]/y^n$) $\mathcal{I}_p = (y_1 + y_2^2 + y_3^3 + y_2^4, y_2^5) \subset \mathcal{O}_p$, of local Hilbert function $H(\mathcal{O}_p/\mathcal{I}_p) = (1, 1, 1, 1, 1)$. Using the computer algebra program MACAULAY [BGS] we calculated its homogenization as $I_3 = (x_1 z + x_2^2 - x_1 x_2, x_1 z^2 + x_2^2 z + x_1^2 z + x_2^3, x_1^3)$, of Hilbert function $H_3 = (1, 3, 5, 5, \dots)$, so $\sigma(\mathfrak{Z}) = 3 < \alpha(\mathfrak{Z}) = 4$. A local dual generator $f' \in L' = (I')^{-1}$ is $f' = Y_2^{[4]} - Y_1 Y_2^{[2]} + Y_1^{[2]} - Y_1 Y_2 - Y_2^{[2]}$. Since \mathcal{I}_p is not homogeneous, its dual generator is determined only up to multiple $\lambda \circ f'$ by a unit λ of R' . We have $L' = \langle f', Y_2^{[3]} - Y_1 Y_2 - Y_1, Y_2^{[2]} - Y_1, Y_2, 1 \rangle$, and, letting $F = \text{Homog}(f', Z, 4)$, we have

$$L_4 = \langle F, X_2^{[3]} Z - X_1 X_2 Z^{[2]} - X_1 Z^{[3]}, X_2^{[2]} Z^{[2]} - X_1 Z^{[3]}, X_2 Z^{[3]}, Z^{[4]} \rangle.$$

Here L_3 contains $f'[3] = \text{Homog}(f', Z, 3) = -X_1 X_2^{[2]} + X_1^{[2]} Z - X_1 X_2 Z - X_2^{[2]} Z$. Note that $Z \cdot_{rp} f'[3] \notin L_4$. Here $L_4 = (L_3 : R_1)$ so L_3 determines L_4 (Proposition 1.13 Equation (1.10)), but $L_4 \neq L_3 \cdot_{rp} Z$ — unlike the simple relation $L_{i+1} = L_i \cdot_{rp} Z$ when $i \geq \alpha(\mathfrak{Z})$. Also $L_4 \neq R_3 \circ (f'[3] \cdot_{rp} Z^{[4]})$. Rather, by Lemma 2.9, we need to use $f'[\alpha]$: thus, $L_j = R_4 \circ (f'[4] \cdot_{rp} Z^{[j]})$.

We now return to one of our themes, deciding when a locally Gorenstein zero-dimensional scheme is arithmetically Gorenstein, with the aid of the inverse system.

Proposition 2.18. CONES THAT ARE AG. *Suppose that $\mathfrak{Z} \subset \mathbb{P}^n$ is a degree- s zero-dimensional locally Gorenstein subscheme, concentrated at a single point $p \in \mathbb{P}^n$. Suppose further that \mathfrak{Z} is defined by a homogeneous ideal \mathcal{I}_p of the local ring \mathcal{O}_p at p (we say that \mathfrak{Z} is conic, see [IK, Lemma 6.1]). Then \mathfrak{Z} is arithmetically Gorenstein.*

Proof. We may suppose that the point is $p = (0 : \dots : 0 : 1)$, and that \mathcal{I}_p is defined by $I' \subset R' = k[y_1, \dots, y_n]$. Then, letting $z = x_{n+1}$ we have $(I_3)_i = \bigoplus_0^i z^{i-a} \cdot I'$, whence it follows that $R/(I_3, z) \cong R'/I'$, implying that R/I_3 is Gorenstein, and that $\Delta H_3 = H(R'/I')$. \square

Remark 2.19. The converse of Proposition 2.18 is false in \mathbb{P}^2 . The ideal $\mathcal{I}_p = (y_1^2, y_1 y_2 - y_2^3) \subset \mathcal{O}_p, p = (0 : 0 : 1)$ has local Hilbert function $H(\mathcal{O}_p/\mathcal{I}_p) = (1, 2, 1, 1, 1)$, and is not homogeneous. The homogenized ideal $I_3 \subset R = k[x_1, x_2, x_3]$ satisfies $I_3 = (x_1^2, x_1 x_2 z - x_2^3, x_2^5)$, of Hilbert function $H_3 = (1, 3, 5, 6, 6, \dots)$. Here z is a non-zero divisor for R/I_3 , and the quotient $R/(z, I_3) \cong k[y_1, y_2]/(y_1^2, y_2^3)$, so \mathfrak{Z} is arithmetically Gorenstein.

D. Bernstein and the second author [BeI], M. Boij and D. Laksov [BoL] gave examples of graded Gorenstein Artin algebras having non-unimodal Hilbert functions. Later M. Boij gave examples of such algebras whose Hilbert functions have arbitrarily many maxima [Bo1]. It follows from Proposition 2.18 that these examples lead to "thick points" that are arithmetically Gorenstein schemes \mathfrak{Z} in \mathbb{P}^n , with non-unimodal h -vector ΔH_3 . We give the first such example of lowest embedding dimension, then socle degree.

Corollary 2.20. *There is an arithmetically Gorenstein, conic, zero-dimensional scheme \mathfrak{Z} concentrated at a single point $p \in \mathbb{P}^5$ with ΔH_3 non-unimodal, and satisfying*

$$\Delta H_3 = (1, 5, 12, 22, 35, 51, 70, 91, 90, 91, \dots, 5, 1). \quad (2.16)$$

The homogeneous form $F' \in \Gamma' = k_{DP}[U, V, W, X, Y]$ defining $\mathcal{I}_p = \text{Ann}(F')$ is $F' = Uf + Vg$ where f, g are general enough degree-15 forms in W, X, Y .

Remark. When \mathfrak{Z} is concentrated at a single point, defined by $\mathcal{I}_p \subset \mathcal{O}_p$, and the local Hilbert function $H(\mathcal{O}_p/\mathcal{I}_p)$ is symmetric, then it is known that \mathcal{I}_p is Gorenstein if and only if the associated graded ideal \mathcal{I}_p^* is also Gorenstein ([Wa, Proposition 1.9], [I2, Proposition 1.7]). It is not hard to show that when $H(\mathcal{O}_p/\mathcal{I}_p)$ is symmetric, \mathfrak{Z} is arithmetically Gorenstein if and only if \mathfrak{Z} is Gorenstein and $\mathcal{I}_p = \mathcal{I}_p^*$.

2.2 Schemes concentrated at an arbitrary point of \mathbb{P}^n .

We now extend the results of the previous subsection to any point $p \in \mathbb{A}^n \subset \mathbb{P}^n$. We translate the point to the origin using the action of the linear group, and use the adjoint representation on Γ , to translate the inverse system. Following F. H. S. Macaulay, we take $\widehat{\Gamma}' = k_{DP}\{\{Y_1, \dots, Y_n\}\}$, the divided power analog of the power series ring, upon which the polynomial ring $R' = k[y_1, \dots, y_n]$ acts by contraction, as before. The rings R, Γ , remain the same, but a finite inverse system will be an R' -submodule of $\widehat{\Gamma}'$ having finite dimension as k -vector space. When $p = (a_1 : \dots : a_n : 1) \in \mathbb{P}^n$, we will sometimes use $q = (a_1, \dots, a_n)$ to specify the point $q = (a_1, \dots, a_n)$ of \mathbb{A}^n without regard to \mathbb{P}^n . We let

$$f_q = (1 - \sum a_i Y_i)^{-1} = 1 + \sum_{k \geq 1} (\sum a_i Y_i)^{[k]} = \sum_k \sum_{U \parallel |U|=k} a^U Y^{[U]}. \quad (2.17)$$

Here f_q is the divided power analog of the exponential series $F_q = \exp(\sum a_i Y_i)$ in the usual power series ring $\widehat{\mathcal{R}}'$. We will sometimes use f_p, F_p to denote the corresponding f_q, F_q .

Lemma 2.21. [Mac, §64, p. 73] INVERSE SYSTEMS FOR IDEALS WITH SUPPORT AN ARBITRARY POINT. *The finite inverse system $J \subset \widehat{\Gamma}'$ (respectively, $J' \subset \widehat{\mathcal{R}'}$ in the differentiation action of R' on $\widehat{\mathcal{R}'}$) is the inverse system of an ideal of R' with support the point $p = (a_1, \dots, a_n) \in \mathbb{A}^n$ if and only if there exists an integer N such that*

$$f_q \subset J \subset \Gamma'_{\leq N} \cdot f_q \subset \widehat{\Gamma}' \quad (2.18)$$

or, respectively,

$$\exp(\sum a_i Y_i) \subset J' \subset \mathcal{R}'_{\leq N} \cdot \exp(\sum a_i Y_i) \subset \widehat{\mathcal{R}'}. \quad (2.19)$$

Proof Outline. Here (2.18) is the divided power analog of (2.19). To show (2.19), note first that $m_q = (y_1 - a_1, \dots, y_n - a_n) \subset R'$ annihilates the one-dimensional vector space $\exp(\sum a_i Y_i) \in \widehat{\mathcal{R}'}$, since y_i acting by differentiation on this series is the same as multiplication by a_i . Likewise, $(m_q^{N+1})^\perp \subset \mathcal{R}'_{\leq N} \cdot \exp(\sum a_i Y_i)$ is immediate, and a dimension check shows (2.19). \square

The following lemma is a consequence of [Mac, §64,66] (see Remark 2.23 below). Given $q = (a_1, \dots, a_n) \in \mathbb{A}^n$ and an ideal J of R' concentrated at the origin, we denote by $T_q(J)$ the translated ideal $T_q(J) = (h(y_1 - a_1, \dots, y_n - a_n) \mid h \in J)$. Clearly, $T_q(J)$ is concentrated at q .

Lemma 2.22. MACAULAY'S COMPARISON LEMMA: CHANGE OF ORIGIN IN \mathbb{A}^n .

- (i) *If $h' \in R'$, f' in $\widehat{\Gamma}'$, then $h'(y_1 - a_1, \dots, y_n - a_n) \circ (f' \cdot f_q) = (h'(y_1, \dots, y_n) \circ f') \cdot f_q$.*
- (ii) *If $L'' \subset \widehat{\Gamma}'$ is the inverse system of an ideal J of R' that is concentrated at the origin, then $L'' \cdot f_q$ is the inverse system of $T_q(J)$.*
- (iii) *Let $\mathcal{I}_q \subset \mathcal{O}_q$ be the ideal $J' \cdot \mathcal{O}_q$, where $J' \subset R'$ has support q . Then the inverse system $L' = (J')^{-1} \subset \widehat{\Gamma}'$ has the form $L' = L'' \cdot f_q$, where $L'' \subset \Gamma'$ is the inverse system of the ideal $J = (T_q)^{-1}(J')$, concentrated at the origin.*
- (iv) *The R' submodules of $\widehat{\Gamma}'$ generated by L' and by L'' in (iii) are isomorphic.*
- (v) *The analogous statements to (i)-(iv) are true for the partial differentiation action of R' on the power series ring $\widehat{\mathcal{R}'}$, with f_q replaced by $F_q = \exp(\sum a_i Y_i)$.*

Proof. The part (ii) is implied by (i). It suffices to show part (i) for monomials h' ; by induction on degree, we may suppose that $h' = y_i$ (since the statement is obvious for $h' = \text{constant}$). Then we have by additivity of contraction, and Lemma 1.2,

$$\begin{aligned} (y_i - a_i) \circ (f' \cdot f_q) &= y_i \circ (f' \cdot f_q) - a_i \circ (f' \cdot f_q) \\ &= (y_i \circ f' \cdot f_q + f' \cdot y_i \circ f_q) - a_i f' \cdot f_q \\ &= y_i \circ f' \cdot f_q + f' \cdot a_i f_q - a_i f' \cdot f_q \\ &= (y_i \circ f') \cdot f_q, \end{aligned}$$

as claimed. This completes the proof of (ii). Any ideal J' of R' concentrated at p satisfies, $J' = T_q(J)$, $J = (T_q)^{-1}(J')$, so (ii) implies (iii). Also, (iv) is immediate. \square

Remark 2.23. Macaulay [Mac, §64, p.72] describes the transform in Lemma 2.22, as follows. Let $F = \sum a_{p_1, \dots, p_n} y_1^{p_1} \dots y_n^{p_n}$ be a polynomial, let $E = \sum c_1^{p_1} \dots c_n^{p_n} (y_1^{p_1} \dots y_n^{p_n})^{-1}$ be a modular equation, and consider the new origin $(-a_1, -a_2, \dots, -a_n)$. The transformed polynomial is $F' = \sum a_{p_1, \dots, p_n} (y_1 - a_1)^{p_1} \dots (y_n - a_n)^{p_n}$, and the transformed modular equation is

$$E' = \sum (c_1 + a_1)^{p_1} \dots (c_n + a_n)^{p_n} (y_1^{p_1} \dots y_n^{p_n})^{-1}.$$

Here the coefficients c are in symbolic notation: that is, after expanding the expressions, $c_1^{p_1} \cdots c_n^{p_n}$ is to be put equal to the coefficient c_{p_1, \dots, p_n} . For $E = 1$, then $E' = \sum a_1^{p_1} \cdots a_n^{p_n} (y_1^{p_1} \cdots y_n^{p_n})^{-1}$, the inverse function of $(x_1 - a_1, \dots, x_n - a_n)$.

Macaulay translates a mutually perpendicular polynomial/inverse system pair at the origin, to one concentrated at the point $q = (a_1, \dots, a_n)$. We rewrite Macaulay's formula for E' using the multiindex $U = (u_1, \dots, u_n)$ where $U \leq P$ means, $u_i \leq p_i$ for each i , as follows:

$$\begin{aligned}
E' &= \sum_{U, P|0 \leq U \leq P} c_{u_1, \dots, u_n} \binom{p_1}{u_1} \cdots \binom{p_n}{u_n} a_1^{p_1 - u_1} \cdots a_n^{p_n - u_n} \cdot (y_1^{p_1} \cdots y_n^{p_n})^{-1} \quad (\text{Macaulay's notation}) \\
&= \sum_{U, P|0 \leq U \leq P} c_{u_1, \dots, u_n} \binom{p_1}{u_1} \cdots \binom{p_n}{u_n} a_1^{p_1 - u_1} \cdots a_n^{p_n - u_n} \cdot Y_1^{[p_1]} \cdots Y_n^{[p_n]} \quad (\text{our notation}) \\
&= \sum_{U, P-U|0 \leq U, 0 \leq P-U} c_{u_1, \dots, u_n} \left(Y_1^{[u_1]} \cdots Y_n^{[u_n]} \right) a_1^{p_1 - u_1} \cdots a_n^{p_n - u_n} \cdot Y_1^{[p_1 - u_1]} \cdots Y_n^{[p_n - u_n]} \\
&= E \cdot f_q.
\end{aligned}$$

The product in the last two steps is that of the divided power ring $\widehat{\Gamma}'$.

Note that our Lemma 2.22, Equation (i) when $h' \circ f' = 0$, is equivalent to Macaulay's formula, so Lemma 2.22 (ii) is a consequence of Macaulay's formula for changing the point of origin.

Fix a point $q = (a_1, \dots, a_n) \in \mathbb{A}^n \subset \mathbb{P}^n$ with projective coordinates $p = (a_1 : \dots : a_n : 1)$. Let $J' \subset R'$ be an ideal supported at q , so that $(R'/J') \cong \mathcal{O}_q/\mathcal{I}_q$, $\mathcal{I}_q = J' \cdot \mathcal{O}_q$ defines an Artin quotient. By Lemma 2.22 its inverse system $L' = (J')^{-1} \subset \widehat{\Gamma}'$ satisfies $L' = L'' \cdot f_q$, where $L'' \subset \Gamma'$ is the inverse system of $J = (T_q)^{-1}(J')$. Recall that $L_p = a_1 X_1 + \cdots + a_n X_n + Z$. Recall the homogenization $\text{Homog}(L'', L_p, u)$ for inverse systems $L'' \subset \Gamma'$ (Definition 2.4).

Theorem 2.24. COMPARISON THEOREM. *Let $I_3 \subset R$ be the saturated ideal defining the scheme \mathfrak{Z} concentrated at the point $p \in \mathbb{A}^n \subset \mathbb{P}^n$, and let $L_3 = I_3^{-1} \subset \Gamma$ be its global inverse system. Let $J' \subset R'$ be the ideal defining $\mathfrak{Z} \subset \mathbb{A}^n$ and $L' = (J')^{-1} \subset \widehat{\Gamma}'$ its affine inverse system. Let $J = T_q^{-1}(J')$, and $L'' = J^{-1} \subset \Gamma'$ its inverse system. Let $\alpha = \alpha(\mathfrak{Z})$ and suppose that $V'' \subset \Gamma'_{\leq \alpha}$ generates L'' (so $L'' = R' \circ V''$), and set $V = \text{Homog}(V'', L_p, \alpha)$.*

(i) *Then the global inverse system L_3 satisfies*

$$(L_3)_i = \text{Homog}(L''_{\leq \alpha}, L_p, i) = R_\alpha \circ (V \cdot {}_{rp} L_p^{[i]}). \quad (2.20)$$

(ii) *Furthermore, let g denote the linear transformation of R taking p to the origin, and g^* the contragradient transform on Γ , and set $\mathfrak{Z}_o = \text{Proj}(R/g(I_3))$, $L_o = (I_{\mathfrak{Z}_o})^{-1}$. Then we have*

$$L = g^* \circ L_o. \quad (2.21)$$

The R -module L_3 is isomorphic to L_o . Also, if \mathfrak{Z} is any zero-dimensional scheme concentrated at p , then $L_3 = (I_{\mathfrak{Z}})^{-1}$ satisfies the first part of (2.20), for a suitable $L'' \subset \Gamma'_{\leq \alpha}$, where $\alpha = \alpha(\mathfrak{Z})$; conversely, if an inverse system L satisfies $L_i = \text{Homog}(L''_{\leq \alpha}, L_p, i)$, then $L = L_3$ for a zero-dimensional scheme \mathfrak{Z} concentrated at p .

Proof. The linear transformation of R taking $p_0 = (0 : \dots : 0 : 1)$ to $p = (a_1 : \dots : a_n : 1)$ is $g(x_1) = x'_1 = x_1 - a_1 z, \dots, g(x_n) = x'_n = x_n - a_n z, g(z) = z' = z$. The contragradient transform of $\Gamma = R^\vee$ satisfies $g^*(v^*)(v) = v^*(g^{-1}v)$, and is readily seen to be $g^*(X_i) = X_i, 1 \leq i \leq n$; and $g^*(Z) = L_p$. The contraction map is equivariant (see, for example [Mac], or [IK, Prop. A3]), so for $h \in R, F \in \Gamma$, $g^*(h \circ F) = g(h) \circ (g^*F)$. Thus, (2.20) follows from Lemma 2.9 and in particular Equation (2.15). The last statement follows from Proposition 2.11 (v), similarly by translation to p . \square

Remark. We believe that equation (2.20) could also be approached directly from Lemma 2.22, using the fact, homogenizing $v \cdot f_q, v \in \Gamma'$ to a given degree, with respect to Z , is the same as homogenizing v with respect to L_p , since $Z^{[j]} \cdot_{rp} \left(1 + \sum_{U|1 \leq |U| \leq j} a^U (X/Z)^{[U]}\right) = L_p^{[j]}$. By Corollary 1.8, $\dim_k V'' = \text{type } \mathcal{O}_3$ is the minimum possible dimension for V'' .

Example 2.25. Let \mathfrak{Z} denote the degree-4 scheme concentrated at $p_1 = (1 : 0 : 1)$, determined by $f' = (Y_1^{[2]} + Y_2^{[2]}) \cdot f_{p_1}$. Then I_3 is the translation to p_1 of $(x_1 x_2, x_1^2 - x_2^2)$, so $I_3 = (x_1 x_2 - z x_2, x_1^2 - 2 z x_2 + z^2 - x_2^2)$, of Hilbert function $H_3 = (1, 3, 4, 4, \dots)$, with $\tau(\mathfrak{Z}) = \alpha(\mathfrak{Z}) = 2$. Here L_3 determines $L = (I_3)^{-1}$, and satisfies, by Theorem 2.24,

$$\begin{aligned} L_3 &= \text{Homog}(V, X_1 + Z, 3), \text{ where } V = \langle X_1^{[2]} + X_2^{[2]}, X_1, X_2, 1 \rangle \\ &= \langle 3X_1^{[3]} + X_1^{[2]}Z + X_1X_2^{[2]} + X_2^{[2]}Z, \langle X_1, X_2 \rangle \cdot (X_1 + Z)^{[2]}, (X_1 + Z)^{[3]} \rangle \end{aligned}$$

Example 2.26. Again, consider the point $p = (1 : 0 : 1) \in \mathbb{P}^2$ and the ideal $\mathcal{I}_p \subset \mathcal{O}_p$ defined by $\mathcal{I}_p = \text{Ann}(f' \cdot f_p)$ where $f' = Y_1^{[2]} + Y_2^{[3]}$: this ideal is the translation to p of the ideal found in Example 2.13, concentrated at $p_0 = (0 : 0 : 1)$. We have $\mathcal{I}_p = ((y_1 - 1)y_2, (y_1 - 1)^2 - y_2^3, (y_1 - 1)^3, y_2^4)$, and its homogenization in $R = k[x_1, x_2, z]$ is $I = ((x_1 - z)x_2, (x_1 - z)^2 z - x_2^3, (x_1 - z)^3, x_2^4)$, of Hilbert function $H_3 = (1, 3, 5, 5, 5, \dots)$, defining a scheme \mathfrak{Z} of regularity $\sigma(\mathfrak{Z}) = 3$. By Theorem 2.24 the inverse system $L = L_3$ is determined by the “generator” element $F = \text{Homog}(f', L_p, 3) = X_1^{[2]} \cdot L_p + X_2^{[3]} \in L$, $L_p = X_1 + Z$: so $L_i = R_3 \circ G_{i+3}, G_{i+3} = F \cdot_{rp} L_p^{[i]}$. Thus we have for L_3 , which determines L ,

$$\begin{aligned} L_3 &= R_3 \circ F \cdot_{rp} (X + Z)^{[3]} = R_3 \circ \left(X_1^{[2]} \cdot (X_1 + Z)^{[4]} + X_2^{[3]} \cdot (X_1 + Z)^{[3]} \right) \\ &= R_3 \circ \left[\left(15X_1^{[6]} + 10X_1^{[5]}Z + 6X_1^{[4]}Z^2 + 3X_1^{[3]}Z^3 + X_1^{[2]}Z^4 \right) + \right. \\ &\quad \left. X_2^{[3]} \cdot \left(X_1^{[3]} + X_1^{[2]}Z + X_1Z^2 + Z^3 \right) \right]. \end{aligned}$$

By the first part of (2.20) in Theorem 2.24, and Example 2.13 this is

$$L_3 = \text{Homog}(V'', L_p, 3), \text{ where } V'' = R' \circ f' = \langle f', Y_2^{[2]}, Y_2, Y_1, 1 \rangle.$$

So $z^3 \circ G_6 = 3X_1^{[3]} + X_1^{[2]}Z + X_2^{[3]} \in L_3$. Note the coefficient 3 on the first term; since $I \circ L = 0$, we have $I \circ (z^3 \circ G_6) = 0$. Thus we have

$$\begin{aligned} (x_1 - z)^3 \circ (z^3 \circ G_6) &= (x_1^3 - 3x_1^2z + 3x_1z^2 - z^3) \circ (3X_1^{[3]} + X_1^{[2]}Z + X_2^{[3]}) \\ &= x_1^3 \circ (3X_1^{[3]}) - 3x_1^2z \circ (X_1^{[2]}Z) + 0 - 0 \\ &= 0 \end{aligned}$$

Proposition 2.27. INVERSE SYSTEM OF A SCHEME CONCENTRATED AT A SINGLE POINT. *Assume that $\text{char } k = 0$, or $\text{char } k > j$. Suppose $\mathfrak{Z} \subset \mathbb{P}$ is a degree- s zero-dimensional scheme concentrated at the point $p = (a_1 : \dots : a_n : 1)$, and regular in degree σ with $\alpha(\mathfrak{Z}) = \alpha$. Then, $W \subset \Gamma$ is the inverse system of \mathfrak{Z} if and only if each of the following equivalent conditions holds:*

- (i) $\exists \alpha \in \mathbb{N} \mid k_{DP}[L_p] \subset W \subset \Gamma_{\leq \alpha} \cdot k_{DP}[L_p]$;
- (ii) (a) $\dim_k W_j = s \ \forall j \geq \tau =_{\text{def}} \sigma - 1$ and
 (b) $\forall (i, n) \mid n \geq \max\{i, \sigma\}, R_{n-i} \circ W_n = W_i$
 (c) $W \subset \Gamma_{\leq \alpha} \cdot k_{DP}[L_p]$;
- (iii) (a) With the conditions (iia) and (iib) above, and $\forall j, L_p^{[j]} \in W_j$, and

$$(b) \quad \forall j, W_j = R_\alpha \circ (W_\alpha \cdot_{rp} L_p^{[j]}).$$

Proof. The condition (iib) above implies the corresponding condition of Proposition 1.13, so (iia),(iib) are equivalent to $I = \text{Ann}(W)$ being the saturated ideal defining a degree- s zero-dimensional scheme $\mathfrak{Z} \subset \mathbb{P}^n$. The third condition (iic) is that of Lemma 2.1, and assures that \mathfrak{Z} has support the point p . The specific bound α arises from the change of coordinates of Lemma 2.22 applied to the formulas $L_j = L_\alpha[j]$ and $L \subset \Gamma_{\leq \alpha} \cdot k[Z]$ – note that $k[Z] = k_{DP}[Z]$ of Proposition 2.11 (ii). Thus, the hypotheses on \mathfrak{Z} imply the first condition (ii) and conversely. That the hypotheses imply (iiib) follows from Theorem 2.24. Evidently, (iiib) implies (iic). \square

2.3 Schemes with finite support.

We now combine the results of previous sections, to determine the inverse system of schemes concentrated at several points. We will assume that coordinates are chosen so that any zero-dimensional scheme \mathfrak{Z} considered lies entirely within the affine chart \mathbb{A}^n where $x_{n+1} \neq 0$. Let p_1, \dots, p_k be distinct points in \mathbb{P} with $x_{n+1} \neq 0$ and $\mathfrak{Z}(u)$ be a degree- s_u zero-dimensional scheme concentrated at p_u with local socle degree $\alpha(u) = \alpha(\mathfrak{Z}(u))$, for $1 \leq u \leq k$ (Definition 2.3). Recall that we denote by $m_{p_u} \subset R$ the homogeneous ideal of $p_u = (a_{u1} : \dots : a_{un} : 1)$, for $1 \leq u \leq k$. Let $\mathfrak{Z} = \bigcup_{u=1}^k \mathfrak{Z}(u)$.

Proposition 2.28. $I_{\mathfrak{Z}}$ is the saturated ideal defining \mathfrak{Z} if and only if

$$m_{p_1} \cap \dots \cap m_{p_k} \supset I_{\mathfrak{Z}} \supset m_{p_1}^{\alpha(1)+1} \cap \dots \cap m_{p_k}^{\alpha(k)+1}. \quad (2.22)$$

The inverse system L is that of such a scheme if and only if it is saturated (Lemma 1.6, equation (1.7)), and

$$\langle L_{p_1}^{[i]}, \dots, L_{p_k}^{[i]} \rangle \subset L_i \subset \langle \Gamma_{\alpha(1)} L_{p_1}^{[i-\alpha(1)]}, \dots, \Gamma_{\alpha(k)} L_{p_k}^{[i-\alpha(k)]} \rangle \quad (2.23)$$

Proof. The condition (2.22) is the condition for the primary decomposition of $I_{\mathfrak{Z}}$ to have p_1, \dots, p_k as the associated points; the condition (2.23) is its translation by (2.2) (see also Lemma 2.1). \square

Further, let $I(1), \dots, I(k)$ be saturated ideals defining $\mathfrak{Z}(1), \dots, \mathfrak{Z}(k)$ and let $L, L(1), \dots, L(k)$ be the global inverse system of $I_{\mathfrak{Z}}, I(1), \dots, I(k)$ respectively. Recall that $\tau(\mathfrak{Z}) = \sigma(\mathfrak{Z}) - 1$.

Theorem 2.29. DECOMPOSITION OF THE INVERSE SYSTEM OF A PUNCTUAL SCHEME. *With \mathfrak{Z} , $\mathfrak{Z}(u)$ and $\alpha(u)$ ($1 \leq u \leq k$) as above, we denote the regularity degree of \mathfrak{Z} by σ , and that of each $\mathfrak{Z}(u)$ by $\sigma(u)$ for $1 \leq u \leq k$ and set $\mathfrak{Z}'(u) = \text{Proj}(R/(m_{p_u}^{\alpha(u)+1} \cap (I(1) \cap \dots \cap \widehat{I(u)} \cap \dots \cap I(k))))$. Then we have,*

- (i) $L = L(1) + \dots + L(k)$.
- (ii) When $i \geq \sigma - 1$, then $L_i = L(1)_i \oplus \dots \oplus L(k)_i$, and $I(1)_i, \dots, I(k)_i$ intersect properly in R_i .
- (iii) $L(u)_i \subset L_i \cap (\Gamma_{\alpha(u)} \cdot L_{p_u}^{[i-\alpha(u)]})$, with equality for $i \geq \min\{i \mid \dim(L_i \cap (\Gamma_{\alpha(u)} \cdot L_{p_u}^{[i-\alpha(u)]})) = s_u\}$. Certainly there is equality for $i \geq \tau(\mathfrak{Z}'(u))$. Also $L(u)_i = R_{j-i} \circ L(u)_j$ if $i \leq j$ and $j \geq \sigma(u)$.

Proof. First, (i) follows from the exactness of the action of R_i on Γ_i ; the perpendicular space in Γ_i to an intersection $I(1)_i \cap \dots \cap I(k)_i$ is the sum $L(1)_i + \dots + L(k)_i$. That the sum is direct when $i \geq \sigma - 1$ arises from $H(R/I_{\mathfrak{Z}})_i = s = \sum_u s_u = \sum_u H(R/I_{\mathfrak{Z}(u)})_i$ when $i \geq \sigma - 1$ and this shows the first statement of (ii), which is equivalent by duality to the second. The inclusion in (iii) arises from the inclusion $I(u) \supset I_{\mathfrak{Z}} + M(p_u)^{\alpha(u)+1}$ by duality, using Equation (2.2). When $i \geq \tau(\mathfrak{Z}'(u))$ we have that $(M(p_u)^{\alpha(u)+1})_i$ and $(I(1) \cap \dots \cap \widehat{I(u)} \cap \dots \cap I(k))_i$ intersect properly

in R_i by (ii), whence it is not hard to show $I(u)_i = (I_3 + M(p_u)^{\alpha(u)+1})_i$. Here is a proof: let $L'(u) = L(1) + \cdots + \widehat{L(u)} + \cdots + L(k)$. Then

$$L_i \cap (\Gamma_{\alpha(u)} \cdot L_{p_u}^{[i-\alpha(u)]}) = (L'(u) + L(u))_i \cap (\Gamma_{\alpha(u)} \cdot L_{p_u}^{[i-\alpha(u)]}) = (L(u)_i + K_i), \quad (2.24)$$

where, when $i \geq \tau$ we may assume $K_i \subset L'(u)_i$, since the sum $L'(u)_i + L(u)_i$ is then direct; but when $i \geq \tau(\mathfrak{Z}'(u))$ we must have $K_i = 0$. The last statement follows from Lemma 1.6. \square

Remark 2.30. Let the degree s of a zero-dimensional scheme $\mathfrak{Z} \subset \mathbb{P}^n$ be given, also an upper bound $N \geq \sigma(\mathfrak{Z})$ on the regularity degree. Suppose that we can calculate the inverse system $(L_3)_i$ in any degree. We may find the primary decomposition of I_3 as follows: first, determine the points p_u of support by testing which powers $L_{p_u}^{[N]} \in (L_3)_N$. Following Theorem 2.29 (iii), then choose $i \geq s + \dim_{\mathbf{k}}(R/m^{s+1})$ and form the intersection $L(u)_i = (L_3)_i \cap (\Gamma_s \cdot L_{p_u}^{[i-s]})$, from which $I(u)$ can be determined (see Example 3.17). However, this may require working in a high degree. Theorem 2.29 (ii) shows that $(L_3)_N$ contains $L(u)_N$ as a direct summand: can we determine $L(u)_N$ from $(L_3)_N$?

Remark 2.31. *Determining when \mathfrak{Z} is arithmetically Gorenstein.* A Gorenstein Artin local algebra has a unique minimum length ideal, its socle, of dimension one as \mathbf{k} -vector space. Thus if \mathfrak{Z} is a zero-dimensional locally Gorenstein scheme in \mathbb{P}^n , each irreducible component \mathfrak{Z}_i has a unique proper subscheme of degree one less than \mathfrak{Z}_i and we denote by \mathfrak{Z}'_i its union with the remaining components. To use the Cayley-Bacharach (CB) criterion (Example 2.14) for a Gorenstein zero-dimensional scheme with k irreducible components, one needs to check the Hilbert function for the k different subschemes $\mathfrak{Z}'_1, \dots, \mathfrak{Z}'_k$: the CB criterion is that (with $\tau = \tau(\mathfrak{Z})$)

$$H(R/I_{\mathfrak{Z}'_i})_{\tau-1} = H(R/I_3)_{\tau-1} \text{ for each } \mathfrak{Z}'_i, i = 1, \dots, k. \quad (2.25)$$

We have seen in Example 2.14 that when \mathfrak{Z} is local, not *conic*, but \mathfrak{Z}' is conic, then \mathfrak{Z} fails the CB criterion. Since being arithmetically Gorenstein is a global property, there are no local criteria for it. Nevertheless, the above equation (2.25), or even the inverse system can be used to check the CB criterion, as we illustrate in the next example.

Example 2.32. *Non AG scheme.* Suppose $R = \mathbf{k}[x_1, x_2, x_3, z]$, and $\Gamma = \mathbf{k}_{DP}[X_1, X_2, X_3, Z]$, let $I_3 = m_p \cap I(2)$, where $m_p = (x_1 - z, x_2 - z, x_3 - z)$, the maximal ideal at $p = (1, 1, 1, 1)$, and $I(2) = (x_1, x_2^2, x_3^2)$, a complete intersection concentrated at $p_0 = (0 : 0 : 0 : 1)$. Then $\mathfrak{Z} = \mathfrak{Z}(1) \cup \mathfrak{Z}(2) \subset \mathbb{P}^3$ with $\mathfrak{Z}(1) = p$, $\mathfrak{Z}(2) = \text{Proj}(R/I(2))$, and

$$I = I_3 = (x_1^2 - x_1x_3, x_1x_2 - x_1x_3, x_1x_3 - x_1z, x_2^2 - x_1z, x_3^2 - x_1z)$$

$H_3 = (1, 4, 5, 5, \dots)$, with $\tau(\mathfrak{Z}) = 2$. A calculation shows $\mathfrak{Z}'(2) = \text{Proj}(R/I'(2))$, where $I'(2) = m_p \cap (I(2))$, has Hilbert function $H'(2) = (1, 4, 4, \dots)$, satisfying the criterion, but $\mathfrak{Z}'(1) = \mathfrak{Z}(2)$, of Hilbert function $H'(1) = H(2) = (1, 3, 4, \dots)$, so \mathfrak{Z} is not arithmetically Gorenstein.

This can be seen using the inverse systems as follows: taking $W = L_3 = (I_3)^{-1}$, $W(1) = L_{\mathfrak{Z}(1)}$, $W(2) = L_{\mathfrak{Z}(2)}$, $L_p = X_1 + X_2 + X_3 + Z$, we have

$$W_j = W(1)_j + W(2)_j = L_p^{[j]} + \langle X_2Z^{[j-1]}, X_3Z^{[j-1]}, X_2X_3Z^{[j-2]}, Z^{[j]} \rangle.$$

The inverse system $W'(2)$ to $I_{\mathfrak{Z}'(2)}$ is obtained by removing from W_j the *generator* $X_2X_3Z^{[j-2]}$ of $W(2)$, not affecting $\dim_{\mathbf{k}} W'(2)_1 = 4$. The dual module $W'(1)$ to $I'_{\mathfrak{Z}'(1)}$ is obtained by removing from W_j the *generator* $L_p^{[j]}$, of $W(1)$ which gives $\dim_{\mathbf{k}} W'(1)_1 = 3$, not 4, as required by the Cayley-Bacharach criterion (2.25).

Remark 2.33. Regularity degree. When \mathfrak{Z} is concentrated at a single point we showed that the regularity and local socle degree are related by $\sigma(\mathfrak{Z}) \leq \alpha(\mathfrak{Z}) + 1$ (see Proposition 2.11 (iv)). This result cannot extend to arbitrary zero-dimensional schemes. When the degree- s scheme \mathfrak{Z} is smooth, we have $\alpha(\mathfrak{Z}) = 0$, but $H_{\mathfrak{Z}}$ can be any sequence such that $\Delta H_{\mathfrak{Z}}$ is an O -sequence of length- s , by Theorem 1.12 (iv). Since for a punctual scheme \mathfrak{Z} , $\sigma(\mathfrak{Z}) = 1 + \tau(\mathfrak{Z})$, with $\tau(\mathfrak{Z}) = \max\{i \mid (\Delta H_{\mathfrak{Z}})_i \neq 0\}$, the maximum regularity degree is s , which occurs when $\Delta H_{\mathfrak{Z}} = (1, 1, \dots, 1)$. Even the degree τ component of the ideal $I_{\mathfrak{Z}}$ or of the inverse system $I_{\mathfrak{Z}}^{-1}$ may be far from determining the support of \mathfrak{Z} . For example, if $s = 2$, the smooth scheme $\mathfrak{Z} = (1 : 0 : 1) \cup (0 : 0 : 1) \subset \mathbb{P}^2$ has inverse system I^{-1} satisfying $(I^{-1})_i = (I_{\mathfrak{Z}}^{-1})_i = \langle Z^{[i]}, (X + Z)^{[i]} \rangle$, $\Delta H_{\mathfrak{Z}} = (1, 1)$, so $\tau(\mathfrak{Z}) = 1$. But the degree- τ component of the inverse system, $(I^{-1})_1 = \langle Z, X + Z \rangle$, only restricts the two points of \mathfrak{Z} to lie on the line $y = 0$.

There has been much study of regularity questions for zero-dimensional schemes. For example M. Chardin and P. Philippon show that if there are forms f_1, \dots, f_n of degrees d_1, \dots, d_n in \mathbb{P}^n , such that $f_1 = \dots = f_n = 0$ contains \mathfrak{Z} , and they form a local complete intersection (LCI) at each support point of \mathfrak{Z} , then the regularity degree of \mathfrak{Z} is at most $d_1 + \dots + d_n - n$ [CharP, Theorem A]. LCI schemes \mathfrak{Z} occur naturally in both singularity theory (see [Mi]) and also in the study of certain hyperplane arrangements (see [Schk]). It could be of interest to explore such zero-dimensional schemes from an inverse system point of view. However, to detect CI or LCI from the inverse system is not so easy, and it is rather simpler to detect if \mathfrak{Z} is Gorenstein.

We give the following basic result bounding the regularity degree of \mathfrak{Z} in terms of the socle degrees of the irreducible components of \mathfrak{Z} , when the number of components is small. We say that k points in \mathbb{P}^n are in (linearly) general position if each subset of s points spans a \mathbb{P}^{s-1} , for $s \leq n + 1$.

Proposition 2.34. *Let \mathfrak{Z} be a zero-dimensional scheme, supported at $p(1), \dots, p(k) \subset \mathbb{P}^n$, and suppose the socle degrees of the irreducible components $\mathfrak{Z}(1), \dots, \mathfrak{Z}(k)$ are $\alpha(1) \leq \dots \leq \alpha(k)$. If $k \leq n + 2$ and the k points are in linearly general position, then the regularity degree $\sigma(\mathfrak{Z})$ satisfies*

$$\sigma(\mathfrak{Z}) \leq \alpha(k) + \alpha(k - 1) + 2. \quad (2.26)$$

Proof. The inverse system $W \subset \mathcal{R}$ for $m_{p(1)}^{\alpha(1)+1} \cap \dots \cap m_{p(k)}^{\alpha(k)+1}$ satisfies, by the partial derivative action of R on \mathcal{R} analogue (2.2), $W_i = \langle L(1)^{[i-\alpha(1)]}, \dots, L(k)^{[i-\alpha(k)]} \rangle_i$. The hypothesis that the points are in linearly general position, implies that the ideal $(L(1)^{[i-\alpha(1)]}, \dots, L(k)^{[i-\alpha(k)]})$ is a complete intersection when $k \leq n + 1$, and an almost complete intersection when $k = n + 2$. Using the Hilbert function of CI's, or a result of R. Stanley (see [I4, Lemma C]) when $k = n + 2$, we have $\dim_k W_i = \sum_u \dim_k R_{\alpha(u)}$ if and only if $i < i - \alpha(k) + i - \alpha(k - 1)$, or $i \geq \alpha(k) + \alpha(k - 1) + 1$; for such i the sum $(L_{\mathfrak{Z}(1)})_i + \dots + (L_{\mathfrak{Z}(k)})_i$ is direct, since each $L_{\mathfrak{Z}(u)}$ has $\tau(\mathfrak{Z}(u)) \leq \alpha(u)$, and we have $\tau(\mathfrak{Z}) \leq \alpha(k) + \alpha(k - 1) + 1$, implying (2.26). \square

Analogous inequalities when $k \geq n + 3$ can be shown in some special cases, with the hypothesis that the points of support are *generic*. However, the general problem of bounding $\sigma(\mathfrak{Z})$ in terms of the $\alpha(u)$ is equivalent to the interpolation problem, of determining the Hilbert function of higher order vanishing ideals at the k points. This problem is open in general, unless $\alpha(u) \leq 2$, or $k \leq n + 2$ (see [AlH, Cha1, Cha2, I3]). When $k = 6$ points on \mathbb{P}^3 , there is exceptional behavior; calculation for $\alpha = 3, 4, \dots$ shows that if $\mathfrak{Z}(u) = \text{Proj } (R/m_{p(u)}^{\alpha+1})$, $u = 1, \dots, 6$, then $\sigma(\mathfrak{Z}) = 2\alpha + 3$.

3 When can we recover the scheme \mathfrak{Z} from a dual form F ?

When can a zero-dimensional scheme $\mathfrak{Z} \subset \mathbb{P}^n$ be recovered from a general element F in $\Gamma_j = (I_{\mathfrak{Z}})_j^{-1}$, the degree- j component of its inverse system? We begin with two examples, first of the

scheme $\text{Proj}(R/m_p^2)$, which cannot be so recovered, and, second, of a non-CM scheme \mathfrak{Z} — having components of different dimension — that can be recovered. We then restate and prove our main result, giving a sufficient condition when $\dim \mathfrak{Z} = 0$ (Theorem 3.3). We give several improvements in special cases, and Corollary 3.11, a consequence concerning subfamilies of the parameter space $\mathbb{P}\mathbf{GOR}(T)$. In Section 3.2 we briefly describe linkage as viewed through the lens of inverse systems, and in Section 3.3 we interpret our results in terms of generalized additive decompositions of forms (Theorems 3.21 and 3.22).

The following example is similar to [IK, Example 5.10].

Example 3.1. *Non-recoverable scheme.* On \mathbb{P}^2 with coordinate ring $R = \mathbf{k}[x, y, z]$, consider the non-Gorenstein ideal $m_p^2, p = (0 : 0 : 1)$ which defines a degree-3 subscheme $\mathfrak{Z} \subset \mathbb{P}^2$. Here $I = I_{\mathfrak{Z}} = (x^2, xy, y^2) \subset R = \mathbf{k}[x, y, z]$, of Hilbert function $H(R/I_{\mathfrak{Z}}) = (1, 3, 3, \dots)$, and local Hilbert function $H(R'/I') = (1, 2)$. Thus $\tau(\mathfrak{Z}) = 1$, $\sigma(\mathfrak{Z}) = 2$, and we have

$$(I_{\mathfrak{Z}})_i^{-1} \cap \Gamma_i = \{Z^{[i]}, Z^{[i-1]}X, Z^{[i-1]}Y\}. \quad (3.1)$$

Taking a general element $F = \alpha Z^{[j]} + \beta XZ^{[j-1]} + \kappa YZ^{[j-1]}$, we find that $\text{Ann}(F)$ contains $\kappa x - \beta y$, so we cannot recover the ideal $I_{\mathfrak{Z}}$ from a single form F . However, we can recover $I_{\mathfrak{Z}}$ using two forms F, G , so from a level algebra of type 2.

Example 3.2. *Line with embedded point.* Let $R = \mathbf{k}[x, y, z], \Gamma = \mathbf{k}_{DP}[X, Y, Z]$. Consider $F = XZ^{[3]} + Y^{[3]}Z \in \Gamma_4$. Then $\text{Ann}(F) = (x^2, xy, xz^2 - y^3, z^4)$ defines an Artin algebra $R/\text{Ann}(F)$ of Hilbert function $T = (1, 3, 4, 3, 1)$. However, $\text{Ann}(F)_{\leq 2} = (x^2, xy)$, defines a scheme $\mathfrak{Z} \subset \mathbb{P}^2$ consisting of a line with an embedded point, whose Hilbert function satisfies $H_{\mathfrak{Z}} = (1, 3, 4, 5, 6, \dots)$.

Taking instead $F_1 = XYZ^{[2]}$, we find $\text{Ann}(F_1) = (x^2, y^2, z^3)$, also of Hilbert function T , and $\text{Ann}(F)_{\leq 2}$ defines a degree-4 scheme $x^2 = y^2 = 0$. More generally let $\mathfrak{Z}_1 = \text{Proj}(R/(g, h))$ be any complete intersection scheme concentrated at $p_0 \in \mathbb{P}^2$, of local Hilbert function $H(R'/(g, h)) = (1, 2, 1)$, and let $f_1 \in \Gamma'$ be a generator of the local inverse system $F = f_1 Z^{[2]}$. Then it is easy to see directly or by Corollary 3.4 that $\text{Ann}(F)_{\leq 2} = (g, h)$, so determines \mathfrak{Z}_1 .

Remark. NONEXISTENCE OF A MORPHISM FROM $\mathbf{GOR}(T)$ TO THE HILBERT SCHEME OF POINTS. Example 3.2 shows that when $T = (1, 3, 4, 3, 1)$, it is not possible to define a morphism from all of $\mathbb{P}\mathbf{GOR}(T)$ (the family of Gorenstein ideals of Hilbert function T , see Definition 3.10 below) to the punctual Hilbert scheme $\mathbf{Hilb}^4(\mathbb{P}^2)$ parametrizing degree-4 zero-dimensional subschemes of \mathbb{P}^2 . The above example also answers negatively a question asked in [IK, p. 142], whether \mathfrak{Z} locally Gorenstein might be a necessary condition for $I_{\mathfrak{Z}}$ to occur as the ideal generated by the lower degree generators of a Gorenstein Artin quotient of $R/I_{\mathfrak{Z}}$ — as here \mathfrak{Z} is not even Cohen-Macaulay. The question of which \mathfrak{Z} occur is open, even when \mathfrak{Z} is restricted to be pure zero-dimensional. See [IK, Remark 5.73 and Chapter 6] for further discussion.

3.1 Recovering \mathfrak{Z} : main results.

We now show our main result about recovering the scheme \mathfrak{Z} from a general element $F \in L_{\mathfrak{Z}}$. Recall that for a zero-dimensional degree- s scheme $\mathfrak{Z} \subset \mathbb{P}^n$ we denote by $\tau(\mathfrak{Z}) = \sigma(\mathfrak{Z}) - 1 = \min\{i \mid (H_{\mathfrak{Z}})_i = s\}$. We denote by $\alpha(\mathfrak{Z})$ the maximum local socle degree of a component of \mathfrak{Z} (see Definition 2.3). We let $\beta(\mathfrak{Z}) = \tau(\mathfrak{Z}) + \max\{\tau(\mathfrak{Z}), \alpha(\mathfrak{Z})\}$, and $L_{\mathfrak{Z}} = (I_{\mathfrak{Z}})^{-1}$. It is evident that for any $F \in (L_{\mathfrak{Z}})_j$, we have $I_{\mathfrak{Z}} \subset \text{Ann}(F)$. We assumed throughout the paper that $\text{char } \mathbf{k} = 0$, or $\text{char } \mathbf{k} = p > j$, where j is the maximum degree of any form considered, here the degree of F (see Example 2.2 for the necessity of this assumption). We assumed that \mathbf{k} is algebraically closed in order for the support of \mathfrak{Z} to consist of \mathbf{k} -rational points. The sequence $\text{Sym}(H_{\mathfrak{Z}}, j)$ is defined in equation (1.1).

Theorem 3.3. RECOVERING THE SCHEME \mathfrak{Z} FROM A GORENSTEIN ARTIN QUOTIENT. *Let \mathfrak{Z} be a locally Gorenstein zero-dimensional subscheme of \mathbb{P}^n over an algebraically closed field \mathbf{k} , $\text{char } \mathbf{k} = 0$ or $\text{char } \mathbf{k} > j$, and let $L_{\mathfrak{Z}} = (I_{\mathfrak{Z}})^{-1}$. Then we have*

- (i) *If $j \geq \beta(\mathfrak{Z})$, and F is a general enough element of $(L_{\mathfrak{Z}})_j$, then $H(R/\text{Ann}(F)) = \text{Sym}(H_{\mathfrak{Z}}, j)$.*
- (ii) *If $j \geq \beta(\mathfrak{Z})$, and F is a general enough element of $(L_{\mathfrak{Z}})_j$, then for i satisfying $\tau(\mathfrak{Z}) \leq i \leq j - \alpha(\mathfrak{Z})$ we have $\text{Ann}(F)_i = (I_{\mathfrak{Z}})_i$. Equivalently, we have $R_{j-i} \circ F = (L_{\mathfrak{Z}})_i$.*
- (iii) *If $j \geq \max\{\beta(\mathfrak{Z}), 2\tau(\mathfrak{Z}) + 1\}$, and $F \in (L_{\mathfrak{Z}})_j$ is general enough, then $\text{Ann}(F)$ determines \mathfrak{Z} uniquely. If $I_{\mathfrak{Z}}$ is generated in degree $\tau(\mathfrak{Z})$, then $j \geq \max\{\beta(\mathfrak{Z}), 2\tau(\mathfrak{Z})\}$ suffices.*

Proof. Since $H(R/\text{Ann}(F))$ is symmetric about $j/2$, (i) follows immediately from (ii). We now show (ii). Suppose first that \mathfrak{Z} has support the single point $p_0 = (0 : \cdots : 0 : 1)$. Let f' of degree $\alpha = \alpha(\mathfrak{Z})$ generate the local inverse system at p_0 of \mathfrak{Z} , let $f = \text{Homog}(f', Z, \alpha)$, and let $L = L_{\mathfrak{Z}}$. Lemma 2.9 shows that $\forall i, L_i = R_{\alpha} \circ (f \cdot_{rp} Z^{[i]})$. Taking $G = f \cdot_{rp} Z^{[j-\alpha]}$, we have $G \in L_j$, and for $i' \geq \alpha$, we have by Proposition 2.11 (ii)

$$R_{i'} \circ G = R_{i'-\alpha} \circ (R_{\alpha} \circ G) = R_{i'-\alpha} \circ L_{j-\alpha} = L_{j-i'}. \quad (3.2)$$

Taking $F = G$, this proves (ii) in this case. Next, if \mathfrak{Z} has support an arbitrary single point $p \in \mathbb{P}^n$, the proof of (ii) is similar, using Theorem 2.24 and (2.20).

Next, suppose that \mathfrak{Z} has degree s , and support $p(1), \dots, p(k)$; thus $I_{\mathfrak{Z}} = I(1) \cap \cdots \cap I(k)$ with $I(u)$ being the ideal of R defining a scheme $\mathfrak{Z}(u)$ having degree s_u , and concentrated at the point $p(u)$, with $\sum s_u = s$. Suppose that $\mathfrak{Z}(u) \subset \mathbb{A}^n$ is defined by $I'(u) \subset R'$ whose inverse system has generator $f'(u)$ (since $I'(u)$ is Gorenstein) in the sense $I'(u)^{-1} = (R' \circ f'(u)) \cdot f_{q(u)}$ where if $p(u) = (a_1(u) : \cdots : a_n(u) : 1)$, we denote by $q(u) = (a_1(u), \dots, a_n(u))$ the coordinates of $p(u)$ in \mathbb{A}^n . Let $G(1), \dots, G(k)$ in Γ_j be the homogenizations $G(u) = \text{Homog}(f'(u), L_{p(u)}, j)$ (see Definition 2.4). Suppose that $i \geq \tau(\mathfrak{Z})$. Denote by \overline{h} the class of $h \bmod I_{\mathfrak{Z}}$, and similarly for ideals, and let $V(u) = I(1) \cap \cdots \cap \widehat{I(u)} \cap \cdots \cap I(k)$. We show first

Claim. For each $u, 1 \leq u \leq k$ we have

$$(\overline{I(u)})_i \oplus (\overline{I(1)} \cap \cdots \cap \widehat{I(u)} \cap \cdots \cap \overline{I(k)})_i = R_i / (I_{\mathfrak{Z}})_i. \quad (3.3)$$

Furthermore, if $i \geq \tau(\mathfrak{Z})$, then $\text{codim } I(u)_i = s_u$ in R_i , and $\dim_{\mathbf{k}} \overline{V(u)}_i = s_u$, and also the codimension of $V(u)_i$ in R_i satisfies $\text{codim } V(u)_i = s - s_u$.

Proof of Claim. That the sum in (3.3) is direct is immediate, since the intersection of the two summands is $(I_{\mathfrak{Z}})_i$. Since $\mathfrak{Z}(u)$ has degree s_u , $\overline{I(u)}_i$ has codimension no greater than s_u in $R_i / (I_{\mathfrak{Z}})_i$. Likewise the vector space $V(u)_i$ has codimension in R_i at most $(\sum_{v \neq u} s_v) = s - s_u$ and likewise, $\overline{V(u)}_i$ has codimension at most $s - s_u$ in $R_i / (I_{\mathfrak{Z}})_i$. Since $i \geq \tau(\mathfrak{Z})$ we have $\dim_{\mathbf{k}}(R_i / (I_{\mathfrak{Z}})_i) = s$, thus we have likewise $\dim_{\mathbf{k}}(R_i / (I(u))_i) = s_u$, $\dim_{\mathbf{k}} R_i / V(u)_i = s - s_u$. And this shows the equality of the Claim. \square

Now let $F = \lambda_1 \cdot G'(1) + \cdots + \lambda_k \cdot G'(k)$, where $G'(u) \in W(u)_j$, $W(u) = I(u)^{-1}$ satisfies (3.2), with G, W there replaced by $G'(u), W(u)$, where $\lambda_u \in \mathbf{k}$ and each $\lambda_u \neq 0$. Consider $w = h \circ G'(u)$, $h \in R_{i'}$. By applying (3.3), we conclude that $h = h' + h''$, $h' \circ G'(u) = 0$, $h'' \in V(u)$, thus $h \circ G'(u) = h'' \circ G'(u) = h'' \circ F$. Thus, we have if $i' \geq \tau$, then $R_{i'} \circ F \supset R_{i'} \circ G'(u)$. Since evidently $R_{i'} \circ F \subset (R_{i'} \circ G'(1) + \cdots + R_{i'} \circ G'(k))$ there is for $i' \geq \tau$ an equality of vector spaces

$$R_{i'} \circ F = (R_{i'} \circ G'(1) + \cdots + R_{i'} \circ G'(k)). \quad (3.4)$$

If we take $i' \geq \max\{\tau(\mathfrak{Z}), \alpha(\mathfrak{Z})\}$, we may take $G'(u) = G(u)$ and apply (3.2) to each term $G'(u)$ of (3.4), and conclude, letting $W(u) = (I_{\mathfrak{Z}(u)})^{-1}$, and taking F as above, $i = j - i'$

$$R_{i'} \circ F = W(1)_i + \cdots + W(k)_i \subset W_i \quad (3.5)$$

When $i \geq \tau = \tau(\mathfrak{Z})$, the sum in (3.5) is direct, and the inclusion on the right is an equality. That a particular $F \in W_j$ satisfies $\dim_k R_{j-i} \circ F = s$, the maximum value possible (so there is equality on the right of (3.5)) implies a fortiori that a general element $F \in W_j$ will have the same property. This completes the proof of (ii). If $j \geq \max\{\beta(\mathfrak{Z}), 2\tau(\mathfrak{Z}) + 1\}$, we have that $\text{Ann}(F)_{\sigma(\mathfrak{Z})} = (I_{\mathfrak{Z}})_{\sigma(\mathfrak{Z})}$, so by (3.5), and Theorem 1.12 (ii), F determines \mathfrak{Z} , showing (iii). This completes the proof of Theorem 3.3 \square .

Remark 3.4. A necessary and sufficient condition for $F = \lambda_1 \cdot G(1) + \cdots + \lambda_k \cdot G(k)$ in Theorem 3.3 to be general enough to satisfy the conclusion, is for each $\lambda_1, \dots, \lambda_k$ to be nonzero.

Proof. The sufficiency was just shown, see especially (3.4). For the necessity, note that if we form F' by omitting the term G_i from F then $I(F')_{\leq \tau} = I(\mathfrak{Z}')_{\leq \tau}$ where $\mathfrak{Z}' = \mathfrak{Z} - \mathfrak{Z}_i$. \square

Remark. We have found no counterexample to show that we could not replace β in Theorem 3.3 by some smaller value, $\beta' \geq 2\tau(\mathfrak{Z})$. What is needed is to establish (3.2) for $i' \geq \alpha' = \beta' - \tau$ — for example (3.2) for $i' \geq \tau(\mathfrak{Z})$ would allow us to replace $j \geq \beta(\mathfrak{Z})$ in Theorem 3.3 (ii) by $j \geq 2\tau(\mathfrak{Z})$, and to simply omit $j \geq \beta(\mathfrak{Z})$ from the statement of Theorem 3.3 (iii) (See Corollary 3.7 below). A measure of the specialness of our result, and a hope for improvement, is given by the rather special form of F in (3.4), far from a generic element of $(L_{\mathfrak{Z}})_j$. The special case \mathfrak{Z} smooth of Theorem 3.3 was shown by M. Boij [Bo2], and the cases \mathfrak{Z} smooth or local *conic* by the second author and V. Kanev [IK, Theorem 5.3E, Lemma 6.1].

Recall that an *SI-sequence* $H = (h_0, h_1, \dots, h_{\lfloor j/2 \rfloor}, \dots, h_j = 1)$ with $h_i = h_{j-i}$ for $1 \leq i \leq j$ is one satisfying $h_1 - h_0, h_2 - h_1, \dots, h_t - h_{t-1}, t = \lfloor j/2 \rfloor$ is an *O-sequence* (Theorem 1.12). We let $n = h_1$. We recover the following result of T. Harima, within our limitation on $\text{char } k = 0$, or $\text{char } k > j$.

Corollary 3.5. [Har] *Given an SI sequence H there is a Gorenstein Artin algebra A with Hilbert function $H(A) = H$.*

Proof. By P. Maroscia's result, Theorem 1.12(iv), there is a smooth zero-dimensional scheme $\mathfrak{Z} \subset \mathbf{P}^n$ with h -vector $\Delta(H_{\mathfrak{Z}}) = \Delta H_{\leq j/2}$. Here $\tau_{\mathfrak{Z}} = t$, and $\alpha_{\mathfrak{Z}} = 0$ for a smooth scheme. By Theorem 3.3 a generic $F \in (L_{\mathfrak{Z}})_j, j = 2\tau$ satisfies $H_F = \text{Sym}(H_{\mathfrak{Z}}, j)$, which is H . This completes the proof. \square

J. Migliore and U. Nagel show further, that there is a reduced arithmetically Gorenstein punctual scheme $\mathfrak{Z}' \subset \mathbf{P}^n$, with h -vector $\Delta H_{\mathfrak{Z}'} = H$ [MigN, Theorem 1.1].

The following Corollary, which determines $\beta(\mathfrak{Z})$ in special cases, shows that we indeed recover the previous results of M. Boij and V. Kanev and the second author when \mathfrak{Z} is smooth or conic.

Corollary 3.6. (i) *Let \mathfrak{Z} be supported at a single point p . Then $\tau(\mathfrak{Z}) \leq \alpha(\mathfrak{Z})$, and $\beta(\mathfrak{Z}) = \tau(\mathfrak{Z}) + \alpha(\mathfrak{Z})$. Also, a general $F \in W_j$ determines \mathfrak{Z} if $j \geq \tau(\mathfrak{Z}) + \alpha(\mathfrak{Z}) + 1$, or if both $j = \tau(\mathfrak{Z}) + \alpha(\mathfrak{Z})$ and $I_{\mathfrak{Z}}$ is generated in degrees less or equal $\tau(\mathfrak{Z})$.*

(ii) *Let \mathfrak{Z} also be conic. Then $\tau(\mathfrak{Z}) = \alpha(\mathfrak{Z})$ and $\beta(\mathfrak{Z}) = 2\tau(\mathfrak{Z})$. If instead \mathfrak{Z} is smooth, then $\alpha(\mathfrak{Z}) = 0$, and also $\beta(\mathfrak{Z}) = 2\tau(\mathfrak{Z})$.*

(iii) *In either the conic or smooth case, a general $F \in W_j$ determines \mathfrak{Z} if either $j \geq 2\tau(\mathfrak{Z}) + 1$, or if both $j \geq 2\tau(\mathfrak{Z})$ and $I_{\mathfrak{Z}}$ is generated in degrees less or equal $\tau(\mathfrak{Z})$.*

We now state the Corollary mentioned in the Remark above. We show that if the statements of Theorem 3.3 are true for each component $\mathfrak{Z}(u)$ of \mathfrak{Z} , but with β replaced by $\beta' = \tau(\mathfrak{Z}) + \alpha'$, then they are true for \mathfrak{Z} with β replaced by β' . We let $L(u) = L_{\mathfrak{Z}(u)}$.

Corollary 3.7. *Suppose that $\mathfrak{Z} = \mathfrak{Z}(1) \cup \dots \cup \mathfrak{Z}(k)$, and that there is an integer $\alpha' \geq \tau(\mathfrak{Z})$ for which (3.2) holds for each $\mathfrak{Z}(u)$, $u = 1, \dots, k$, with G, L there replaced by a suitable choice of general enough $G'(u) \in L(u)_j$, with $j = \tau(\mathfrak{Z}) + \alpha'$ and $i' = \alpha'$. Then the conclusions of Theorem 3.3 hold with $\beta(\mathfrak{Z})$ replaced by $\beta' = \tau(\mathfrak{Z}) + \alpha'$.*

Proof. Taking $F = \sum \lambda(u)G'(u)$ after (3.4), the proof is essentially the same (except we no longer take $G'(u) = G(u)$). Since $G'(u)$ is assumed to satisfy (3.2) for $j = \tau(\mathfrak{Z}) + \alpha'$, with $i' = \alpha'$ in place of $i = \alpha$, we obtain the conclusion of Theorem 3.3 (ii) but with $j = \tau(\mathfrak{Z}) + \alpha'$. For larger $j' = j + c$, $c \geq 0$, we note that (3.2) is still satisfied, replacing $G'(u)$ by $G'(u) \cdot {}_{rp}L_p^{[c]} \in \Gamma_{j'}$, and $i' = \alpha'$ by $i' = \alpha' + c$. This implies Theorem 3.3 (ii),(iii), but with β replaced by β' . This complete the proof of Corollary 3.7. \square

Example 3.8. Let $R = k[X_1, X_2, Z]$ and $f = \text{Homog}(f', Z, 4)$ from Example 2.17 where $f' = Y_2^{[4]} - Y_1Y_2^{[2]} + Y_1^{[2]} - Y_1Y_2 - Y_2^{[21]}$. Here \mathfrak{Z} is concentrated at a single point $p_0 = (0 : 0 : 1) \in \mathbb{P}^2$, the Hilbert function $H(R/I_3) = (1, 3, 5, 5, \dots)$, so $\tau(\mathfrak{Z}) = 2, \alpha(\mathfrak{Z}) = 4$, and $\beta(\mathfrak{Z}) = 2 + 4 = 6$. The Corollary 3.6 implies that for $j \geq 6$, a general $F \in L_j, L = (I_3)^{-1}$ has $H_F = \text{Sym}(H_3, j)$. However, a calculation shows that this occurs for a general $F \in L_4$ (see Example 2.17 for L_4), hence for $j \geq 4$. In particular, if F is a general element of L_5 , $H(R/\text{Ann } F) = (1, 3, 5, 5, 3, 1)$, and $\text{Ann } (F)_{\leq 3} = (x_1x_2 - x_2^2 - x_1z, x_2^3 + x_1^2z + x_2^2z + x_1z^2, x_1^3) = (I_3)_{\leq 3}$. Thus F determines \mathfrak{Z} since $\sigma(\mathfrak{Z}) = 3$.

Example 3.9. Consider the subscheme $\mathfrak{Z} = \mathfrak{Z}(1) \cup \mathfrak{Z}(2)$ of \mathbb{P}^2 , with $\mathfrak{Z}(1)$ the scheme of Example 3.8 concentrated at $p_1 = (0 : 0 : 1)$ and $\mathfrak{Z}(2)$ the degree-4 scheme concentrated at $p_2 = (1 : 0 : 1)$, determined by $f' = (Y_1^{[2]} + Y_2^{[2]}) \cdot f_{p_2}$, of Example 2.25, where $\tau(\mathfrak{Z}(2)) = \alpha(\mathfrak{Z}(2)) = 2$. The intersection $I_3 = I_{\mathfrak{Z}(1)} \cap I_{\mathfrak{Z}(2)}$ satisfies (calculated in MACAULAY)

$$I_3 = (x_1^3 + x_1^2x_2 - 2x_1x_2^2 - 2x_1^2z - x_1x_2z + x_2^2z + x_1z^2, \\ x_1^2x_2^2 - 4x_1x_2^3/3 - x_1^2x_2z - x_1x_2^2z + x_2^3z + x_1x_2z^2, x_1x_2^3, x_2^4 - x_1^2x_2z + x_2^3z + x_1x_2z^2),$$

of Hilbert function $H_3 = (1, 3, 6, 9, 9, \dots)$, $\tau(\mathfrak{Z}) = 3, \alpha(\mathfrak{Z}) = 4$. By Corollary 3.7 and the calculation of Example 3.8 for $\mathfrak{Z}(1)$, as well as Corollary 3.6 applied to $\mathfrak{Z}(2)$, we may replace $\beta(\mathfrak{Z}) = \tau(\mathfrak{Z}) + \alpha(\mathfrak{Z}) = 3 + 4$ in Theorem 3.3 for \mathfrak{Z} by $\beta' = 3 + 3 = 6$. Thus, a general $F \in (L_3)_6$ satisfies $H(R/\text{Ann } (F)) = \text{Sym}(H_3, 6) = (1, 3, 6, 9, 6, 3, 1)$.

We now derive some further consequence of our main theorem, along the lines of Lemma 6.1 of [IK], shown there in the special case of \mathfrak{Z} conic or smooth. We introduce first some definitions from [IK]. For $F \in \Gamma_j$ we let $H_F = H(R/\text{Ann } (F))$.

Definition 3.10. A zero-dimensional scheme \mathfrak{Z} is an *annihilating scheme* for $F \in \Gamma$ if $I_3 \subset I_F = \text{Ann } (F)$. An annihilating scheme is *tight* if also $\deg \mathfrak{Z} = \max_i \{(H_F)_i\}$. If $T = (1, h_1, \dots, h_{j-1}, 1)$ is a sequence of integers symmetric about $j/2$ we denote by $\text{PGOR}(T)$ the locally closed subvariety of $\mathbb{P}(\Gamma_j)$ parametrizing forms $F \in \Gamma_j$ — up to constant multiple — such that $H_F = T$. We denote by **PGOR**(T) (in boldface) the corresponding scheme, whose scheme structure is defined by determinantal ideals of certain catalecticant matrices, corresponding to the conditions $(H_F)_u = T_u$ (see [IK]).

K. Ranestad and A. Bernardi [BeRa] point out that the proof of W. Buczyńska and J. Buczyński [BuB, Lemma 2.4] shows that a tight annihilating scheme \mathfrak{Z} must be locally Gorenstein: otherwise

there would be smaller degree locally Gorenstein scheme \mathfrak{Z}' constructed from \mathfrak{Z} apolar to F , contradicting \mathfrak{Z} “tight”. This answers simply a question posed in [IK, Remark p. 142].

The tangent space \mathcal{T}_F to the affine cone over $\mathbf{PGOR}(T)$ at F is isomorphic to $R_j/((\text{Ann } F)^2)_j$ [IK, Theorem 3.9]. We denote by $\nu = \nu(\mathfrak{Z})$ the order $\nu(\mathfrak{Z}) = \min\{i | (H_{\mathfrak{Z}})_i \neq r_i\}$ of $I_{\mathfrak{Z}}$. We denote by $U_{\mathfrak{Z}} \subset \mathbf{PGOR}(T)$, $T = \text{Sym}(H_{\mathfrak{Z}}, j)$ or more precisely by $U_{\mathfrak{Z}}(j)$ the family of $F \in \Gamma_j$, up to constant multiple, such that $F \in (I_{\mathfrak{Z}})_j^\perp$ and $H_F = T$. Evidently $F \in \Gamma_j$ satisfies $F \in U_{\mathfrak{Z}}(j)$ if and only if $\text{Ann } (F)_i = (I_{\mathfrak{Z}})_i$ for $i \leq j/2$ (since $I_{\mathfrak{Z}} \subset \text{Ann } (F)$ when $F \in (I_{\mathfrak{Z}})_j^\perp$). Below, we will usually omit writing *up to constant multiple* when this is clear from the context, or unimportant. The zero-dimensional Hilbert scheme $\mathbf{Hilb}^s(\mathbb{P}^n)$ parametrizes degree- s subschemes of \mathbb{P}^n (see [IK1]).

Corollary 3.11. *Let \mathfrak{Z} be a zero-dimensional degree s locally Gorenstein scheme of \mathbb{P}^n having regularity degree $\sigma(\mathfrak{Z})$, let $j \geq 2\tau(\mathfrak{Z})$, and let $F \in (I_{\mathfrak{Z}})_j^\perp$.*

- (i) *If $j \geq \beta(\mathfrak{Z})$ (or if \mathfrak{Z} satisfies the hypothesis of Corollary 3.7 and $j \geq \beta'(\mathfrak{Z})$), there is an open dense family $F \in (I_{\mathfrak{Z}})_j^\perp$ such that $F \in U_{\mathfrak{Z}}(j)$. For such F , we have $(\text{Ann } (F))_i = (I_{\mathfrak{Z}})_i$ for $i \leq j - \tau(\mathfrak{Z})$, and \mathfrak{Z} is a tight annihilating scheme of F .*
- (ii) *If $j \geq 2\tau(\mathfrak{Z})$, and F satisfies $H_F = \text{Sym}(H_{\mathfrak{Z}}, j)$, and if $Y \subset \mathbb{P}^n$ is any zero-dimensional subscheme satisfying $\deg(Y) \leq s$ and $I_Y \subset \text{Ann}(f)$, then $\deg(Y) = s$ and $(I_Y)_i = (I_{\mathfrak{Z}})_i$ for $i \leq j - \tau(\mathfrak{Z})$.*
- (iii) *If F satisfies $H_F = \text{Sym}(H_{\mathfrak{Z}}, j)$, and if also either*
 - (a) *$j \geq 2\tau(\mathfrak{Z}) + 1$, or*
 - (b) *$j \geq 2\tau(\mathfrak{Z})$, and $((I_{\mathfrak{Z}})_{\leq \tau}) = I_{\mathfrak{Z}}$,**then \mathfrak{Z} is the unique tight annihilating scheme of F .*
- (iv) *If F satisfies $H_F = \text{Sym}(H_{\mathfrak{Z}}, j)$, then $\text{Ann } (F)^2_i = (I_{\mathfrak{Z}}^2)_i$ for $i \leq j - (\tau - \nu)$.*
If also $\tau \leq \nu$ and \mathfrak{Z} is a tight annihilating scheme of F , then the tangent space \mathcal{T}_F to the affine cone over $\mathbf{PGOR}(T)$, $T = \text{Sym}(H_{\mathfrak{Z}}, j)$ at F satisfies

$$\dim_k \mathcal{T}_F = s + \dim_k((I_{\mathfrak{Z}}/(\mathcal{I}_{\mathfrak{Z}})^2)_j).$$

- (v) *If $Y \subset \mathbf{Hilb}^s(\mathbb{P}^n)$ is locally closed, and $\mathfrak{Z}_y, y \in Y$ is the corresponding family of degree s zero-dimensional subschemes of \mathbb{P}^n , if $H(R/\mathcal{I}_{\mathfrak{Z}_y}) = H$ for all $y \in Y$, if $\sigma = \tau + 1$ is the generic regularity degree of $\mathfrak{Z}_y, y \in Y$ (attained for an open subset of Y), and if $j, \mathcal{I}_{\mathfrak{Z}}$ satisfy (iiia) or (iiib) above, and $T = \text{Sym}(H_{\mathfrak{Z}}, j)$, then there exists a subfamily $U_Y \subset \mathbf{PGOR}(T)$ satisfying*
 - (c) *$F \in U_Y \Leftrightarrow H_F = T$ and \mathfrak{Z}_y is a tight annihilating scheme of F ,*
 - (d) *$\dim(U_Y) = \dim(Y) + s - 1$.*

Proof. Here the main assertion (i) follows directly from Theorem 3.3 and the proof of [IK, Lemma 6.1]; we need in (i) the hypothesis $j \geq \beta(\mathfrak{Z})$ in order to use Theorem 3.3. For any $j \geq 2\tau(\mathfrak{Z})$, the assumption $H_F = \text{Sym}(H_{\mathfrak{Z}}, j)$, and that $I_{\mathfrak{Z}} \subset \text{Ann } (F)$ entail most of (ii)-(iv). \square

Example 3.12. Consider the subscheme \mathfrak{Z} of Example 3.9, for which Corollary 3.7 applies for $\beta'(\mathfrak{Z}) = 6$, and choose a general $F \in (I_{\mathfrak{Z}})_6^\perp$; then $T = H_F = \text{Sym}(H_{\mathfrak{Z}}, 6) = (1, 3, 6, 9, 6, 3, 1)$. A calculation shows that $\dim_k R/((I_{\mathfrak{Z}})^2)_6 = 27$. Since $\nu = \tau$ for \mathfrak{Z} , Theorem 3.11 (iv) implies that $\dim_k \mathcal{T}_F = 27$; this is easy to check directly since $\text{Ann } (F)_{\leq 3} = \langle h_3 \rangle$, so $\text{Ann } (F)_6^2 = \langle h_3^2 \rangle$ of codimension 1 in R_6 . Since $r = 3$, $\mathbf{PGOR}(T)$ is smooth; this here corresponds to the smoothability of degree-9 schemes in \mathfrak{Z} . The dimension of $\mathbf{PGOR}(T)$ is 27, since $\dim(\mathbf{Hilb}^9(\mathbb{P}^2)) = 18$, and the dimension of the fiber of $\mathbf{PGOR}(T)$ over $\mathbf{Hilb}^9(\mathbb{P}^2)$ is 9.

Strikingly, if $j = 7$, so $T' = (1, 3, 6, 9, 9, 6, 3, 1)$, the analogous dimension is $\dim_k \mathcal{T}_F = 30$ (since $(\text{Ann}(F)^2)_7 = (I_3)_7^2 = h_3 \cdot (I_3)_4$, of dimension 6); when $j \geq 8$ the dimension is again 27, as can be checked by calculating $H(R/(I_3)^2)$.

3.2 Dualizing module as ideal, and linkage.

We first recall a result of M. Boij, giving conditions under which the dualizing module of \mathfrak{Z} is an ideal of R/I_3 . A consequence of his criterion and Theorem 3.3 is that the dualizing module can always be so realized when \mathfrak{Z} has dimension zero, and is locally Gorenstein (Corollary 3.15). We then give an example to illustrate how the inverse systems behave in linkage.

M. Boij's theorem pertains to d -dimensional Cohen Macaulay rings $B = R/I$, and $d - 1$ dimensional Gorenstein quotients. Let $\kappa(B)$ denote the degree of the polynomial $(1 - z)^d \text{Hilb}_X(z)$: here $\text{Hilb}_X(z)$ is the Hilbert series $\sum H_3(i)z^i$, so $\kappa(B)$ is the highest socle degree of a minimal reduction of B .

Theorem 3.13. (M. Boij [Bo2]) *Let $B = R/I$ be a Cohen-Macaulay algebra of dimension d , and let $J \subset B$ be an ideal of initial degree at least $\kappa(B) + 2$ such that B/J is Gorenstein of dimension $d - 1$.*

Then there is an isomorphism $J \rightarrow \text{Ext}_R^{r-d}(B, R) = \omega_B$, which is homogeneous of degree $-\kappa(B/J) - r + d - 1$.

We consider the special case $d = 1$, and $I = I_3$, the homogeneous defining ideal of a zero-dimensional scheme \mathfrak{Z} . Then Boij's theorem becomes,

Corollary 3.14. *Let \mathfrak{Z} be a zero-dimensional subscheme of \mathbb{P}^n , and let J be an ideal of $B = R/I_3 = \mathcal{O}_{\mathfrak{Z}}$ having initial degree at least $\tau(\mathfrak{Z}) + 2$, such that B/J is Gorenstein of dimension zero and socle degree j . Then there is an isomorphism $J \rightarrow \text{Ext}_R^{r-1}(B, R) = \omega_B$, which is homogeneous of degree $(-j - r)$.*

Our Main Theorem 3.3 and M. Boij's theorem imply

Corollary 3.15. *Let \mathfrak{Z} be a locally Gorenstein zero-dimensional scheme of \mathbb{P}^n . Then there are ideals J of $\mathcal{O}_{\mathfrak{Z}} = R/I_3$ satisfying the conclusions of Corollary 3.14, with $\mathcal{O}_{\mathfrak{Z}}/J$ of socle degree j , provided $j \geq \max\{\beta(\mathfrak{Z}), 2\tau(\mathfrak{Z}) + 1\}$. Any such ideal has the form $J = \text{Ann}(F)/I_3$, $F \in (I_3^{-1})_j \subset \Gamma_j$. Also, if $j \geq 2\tau(\mathfrak{Z}) + 1$ and F is any element of $(I_3^{-1})_j$ such that $H_F = H(R/\text{Ann}(F)) = \text{Sym}(H_3, j)$, then $J = \text{Ann}(F)/I_3$ is isomorphic to the dualizing module of \mathfrak{Z} .*

Proof. The second statement follows from M. Boij's theorem for $B = \mathcal{O}_{\mathfrak{Z}}$, and Macaulay's result connecting the socle of R/J' for an Artinian quotient, and generators of the inverse system of J' (see Corollary 1.8); J' is Gorenstein if and only if J'^{-1} is principal. The third statement follows from Corollary 3.14 and the definition of $\text{Sym}(H_3, j)$ (see Equation (1.1)); the restriction $j \geq 2\tau(\mathfrak{Z}) + 1$ and $H_F = \text{Sym}(H_3, j)$ implies that the order of $\text{Ann}(F)/I_3$ is at least $\tau(\mathfrak{Z}) + 2$, satisfying the hypotheses of M. Boij's theorem, and $\mathcal{O}_{\mathfrak{Z}}/J \cong R/\text{Ann}(F)$, so is Gorenstein. By Theorem 3.3 and Corollary 3.11, such F exist with $H_F = \text{Sym}(H_3, j)$ if $j \geq \beta(\mathfrak{Z})$. \square

M. Boij showed that when \mathfrak{Z} is smooth, then the conclusions of Corollary 3.15 hold also for $j \geq 2\tau(\mathfrak{Z}) - 1$. His work is related to that of M. Kreuzer in [Kr1, Kr2]. Corollary 3.15 can be used as a test of whether a Gorenstein scheme is arithmetically Gorenstein, since \mathfrak{Z} is aG if and only if the dualizing module is principal.

Example 3.16. Let $\mathfrak{Z} = \mathfrak{Z}(1) \cup \mathfrak{Z}(2) \subset \mathbb{P}^3$ be the scheme of Example 2.32, where $\mathfrak{Z}(1) = p$, $p = (1 : 1 : 1 : 1)$ is a smooth point, and $\mathfrak{Z}(2) = \text{Proj}(R/I(2))$ where $I(2) = (x_1, x_2^2, x_3^2)$, is a CI at $p_0 = (0 : 0 : 0 : 1)$. We found there that \mathfrak{Z} was not arithmetically Gorenstein, although $\Delta H_3 = (1, 3, 1)$ is the h -vector of a Gorenstein ideal (it is a *Gorenstein sequence*). Since $\alpha(\mathfrak{Z}) = \tau(\mathfrak{Z}) = 2$, we have

$\beta(\mathfrak{Z}) = \tau(\mathfrak{Z}) + \alpha(\mathfrak{Z}) = 4$. By Corollary 3.15, it suffices to take a general element $F \in (L_{\mathfrak{Z}})_5$, to see the dualizing module as the ideal $J = \text{Ann}(F)/I_{\mathfrak{Z}}$. We have, taking $L_p = X_1 + X_2 + X_3 + Z$

$$(L_{\mathfrak{Z}})_5 = \langle X_2 Z^{[4]}, X_3 Z^{[4]}, X_2 X_3 Z^{[3]}, Z^{[5]}, L_p^{[5]} \rangle.$$

A calculation shows that with $F = X_2 Z^{[4]} + X_3 Z^{[4]} + X_2 X_3 Z^{[3]} + Z^{[5]} + L_p^{[5]}$, we have $H_F = \text{Sym}(H_{\mathfrak{Z}}, 5) = (1, 4, 5, 5, 4, 1)$, and that $\text{Ann}(F) = (I_{\mathfrak{Z}}, x_2 x_3 z^2 - x_2 z^3 - x_3 z^3 + z^4, x_1 z^4 - z^5/2)$. The dualizing module $\text{Ann}(F)/I_{\mathfrak{Z}}$ is not principal, confirming that \mathfrak{Z} is not arithmetically Gorenstein.

We now give an example showing how inverse systems behave under linkage: here $\mathfrak{Z} = \mathfrak{Z}(1) \cup \mathfrak{Z}(2)$ is AG (even CI).

Example 3.17. *Inverse systems of linked local CI's.* We consider inverse systems of three ideals in $R = k[x_1, x_2, z]$ defining punctual subschemes $\mathfrak{Z} = \mathfrak{Z}(1) \cup \mathfrak{Z}(2)$ of \mathbb{P}^2 . The ideal $I(1) = (x_1 - z, x_2) = M(p_1)$ defines the simple point $\mathfrak{Z}(1) = p_1 = (1, 0, 1)$. The ideal $I(2)$, concentrated at $p = (0 : 0 : 1)$ defines a degree 5 scheme $\mathfrak{Z}(2)$, that of Example 2.13, (there termed \mathfrak{Z}), which is a local complete intersection:

$$I(2) = I_{\mathfrak{Z}(2)} = (x_1 x_2, x_1^2 z - x_2^3, x_1^3),$$

of Hilbert function $H_{\mathfrak{Z}(2)} = (1, 3, 5, 5, \dots)$. Their intersection is the ideal $I = (x_1 - z, x_2) \cap I_{\mathfrak{Z}(2)} = (x_1 x_2, x_1^3 + x_2^3 - x_1^2 z)$, a complete intersection defining the degree 6 punctual scheme \mathfrak{Z} , of Hilbert function $H_{\mathfrak{Z}} = (1, 3, 5, 6, 6, \dots)$. Thus $I(1)$ and $I(2)$ determine the two irreducible components of \mathfrak{Z} , which are linked through \mathfrak{Z} . Letting $W = I^{-1}$, $W(1) = I(1)^{-1}$, and $W(2) = I(2)^{-1}$ denote the corresponding inverse systems, we have $W = W(1) + W(2)$, where the sum must be direct in degrees at least $\tau(\mathfrak{Z}) = 3$ by Theorem 2.29 (ii). The inverse system $W(1)$ satisfies $W(1)_i = \langle (X_1 - Z)^{[i]} \rangle$, while $W(2)$ satisfies, from Example 2.13

$$W(2)_i = \langle X_1^{[2]} Z^{[i-2]} + X_2^{[3]} Z^{[i-3]}, X_2^2 Z^{[i-2]}, X_2 Z^{[i-1]}, X_1 Z^{[i-1]}, Z^{[i]} \rangle$$

By the Decomposition Theorem 2.29(i), we have

$$\begin{aligned} W_i &= W(1)_i + W(2)_i \\ &= \langle (X_1 - Z)^{[i]}, X_1^{[2]} Z^{[i-2]} + X_2^{[3]} Z^{[i-3]}, X_2^2 Z^{[i-2]}, X_2 Z^{[i-1]}, X_1 Z^{[i-1]}, Z^{[i]} \rangle. \end{aligned}$$

Note that the above sum is direct in degrees at least three, but not direct in degrees less or equal two, as is evident by regarding $H_{\mathfrak{Z}(1)} = (1, 1, \dots)$ and $H_{\mathfrak{Z}(2)}, H_{\mathfrak{Z}}$. By the Decomposition Theorem 2.29(iii) we have

$$W(2)_i = W_i \cap \langle k[X_1, X_2]_{\leq 3} \cdot {}_{rp} k[Z] \rangle_i,$$

the intersection of W and the inverse system of m_p^4 (here $4 = \alpha(\mathfrak{Z}(2)) + 1$), whenever the dimension of the right side is 5, which occurs for $i \geq 4$.

That \mathfrak{Z} is AG can be seen from the inverse system, following Lemma 1.9, by showing that $L_{\mathfrak{Z}} \cap \Gamma_z = W \cap k_{DP}[X_1, X_2]$ is a principal $R' = k[x_1, x_2]$ -module: in fact, $G = X_1^{[3]} - X_2^{[3]}$ generates this intersection.

Finally, from the properties of linkage, $I(1)/I$ has dualizing module isomorphic to $R/(I(2))$, and conversely $I(2)/I$ has dualizing module $R/I(1)$. In particular the number of generators of $I(1)/I$ (here two) is the same as the dimension of $\text{Soc}(R/I(2))$ and the number of generators of $I(2)/I$ (here one) is $\dim_k \text{Soc}(R/I(1))$. In addition, since R/I is locally Gorenstein, similar properties hold for the localizations at p, p_1 . Here at p_1 , $(R/I(1))_{p_1} \cong R'/m_{p_1}$, has one-dimensional socle, and $m_{p(1)} \cong I_{p_1}$, the localization, so there are zero generators of the quotient. Also, $(R'/I(2))_{p_1} = 0$, so has zero socle, and $I(2)_{p_1} = R'_{p_1}$ has one generator.

3.3 Generalized additive decompositions.

We recall the GAD given in (1.2) for a degree- j form of $\Gamma = \mathbb{k}[X, Y]$, namely

$$F = \sum_i B_i L_i^{[j+1-s_i]}, \deg B_i = s_i - 1, \deg L_i = 1, s = \sum s_i.$$

Each term $B_i L_i^{[j+1-s_i]}$ corresponds to a single support point $p_i : l_i = 0$ of \mathbb{P}^1 , occuring with multiplicity s_i . Our aim is to model this kind of decomposition in $r \geq 3$ variables. The following definition is more general than that of [I3, Def. 4A], but is related to the concept of annihilating scheme introduced there [I3, Def. 4D] (see Definition 3.10 above).

Definition 3.18. For $F \in \Gamma_j$, we say that $F = F_1 + \dots + F_k$ is a *generalized additive decomposition* (GAD) of F , having (total) length $s = \sum s_i$, of partition $\pi = (s_1, \dots, s_k)$, with k parts, associated to the scheme \mathfrak{Z} , if \mathfrak{Z} is a degree- s punctual scheme \mathfrak{Z} whose decomposition into irreducible schemes is $\mathfrak{Z} = \cup \mathfrak{Z}_i$, where $\deg \mathfrak{Z}_i = s_i$, and each $F_i \in I_{\mathfrak{Z}_i}^\perp$ for $i = 1, \dots, k$. We say that a GAD of F is *tight* if \mathfrak{Z} is a tight annihilating scheme of F : namely, if $s = \deg \mathfrak{Z} = \max_i \{(H_F)_i\}$ (Definition 3.10). We say that a GAD is *unique* if the k summands F_1, \dots, F_k are unique.

The form of each term F_i , corresponding to \mathfrak{Z}_i , can be read from Theorem 2.24 or Proposition 2.27; F_i is an element of the degree- j homogenization of the local inverse system of \mathfrak{Z}_i .

Lemma 3.19. *If F has a length- s GAD, then $\forall i \geq 0$, we have $(H_F)_i \leq s$.*

Proof. We have $\text{Ann}(F) \supset \mathcal{I}_{\mathfrak{Z}}$, hence $(H_F)_i \leq (H_{\mathfrak{Z}})_i$, but $(H_{\mathfrak{Z}})_i$ is bounded above by $\deg \mathfrak{Z}$. \square

Which forms F have a length- s GAD? When is the GAD for F unique? Recall that we denote by $\sigma(\mathfrak{Z})$ the regularity degree of \mathfrak{Z} (see Theorem 1.12), and by $\tau(\mathfrak{Z}) = \sigma(\mathfrak{Z}) - 1$. Evidently we have

Lemma 3.20. *If F is annihilated by a zero-dimensional scheme \mathfrak{Z} , $\mathfrak{Z} = \mathfrak{Z}_1 \cup \dots \cup \mathfrak{Z}_k$ as in Definition 3.10, then F has a GAD of length $\leq s$ associated to \mathfrak{Z} . If also $\deg F \geq \tau(\mathfrak{Z})$, then the GAD has length s , is of partition (s_1, \dots, s_k) , $s_i = \deg \mathfrak{Z}_i$, and this GAD is the unique GAD of F that is associated to \mathfrak{Z} .*

Proof. For $j \geq \tau(\mathfrak{Z})$ we have $(H_{\mathfrak{Z}})_j = s$, hence $(I_{\mathfrak{Z}})_j^\perp = (I_{\mathfrak{Z}_1})_j^\perp \oplus \dots \oplus (I_{\mathfrak{Z}_k})_j^\perp$, and the GAD is unique. \square

The following result is an immediate consequence of [IK, Theorem 5.31], and Definition 3.18. It does not extend simply to $r > 3$ (see [Bo3, Theorem 6.42], [ChoI2], and the discussion in [IK, §6.4]).

Theorem 3.21. **UNIQUENESS OF GAD WHEN $r = 3$.** *If $r = 3$ and $H_F \supset (s, s, s)$ then F has a unique tight GAD of length s , up to permutation and change of scale, and no GAD's of smaller length than s .*

Proof. By Theorem 5.31 of [IK], F has a unique tight annihilating scheme \mathfrak{Z} ; this determines a unique GAD by Lemma 3.20, since $j = \deg F > \sigma(\mathfrak{Z})$ (as here we have $j \geq 2\sigma(\mathfrak{Z})$). \square

Recall from Definition 2.3 that $\alpha(\mathfrak{Z})$ is the highest socle degree of a component of \mathfrak{Z} . Finally we have,

Theorem 3.22. *Suppose that \mathfrak{Z} is a Gorenstein zero-dimensional subscheme of \mathbb{P}^n , and that $j \geq \max\{\tau(\mathfrak{Z}) + \alpha(\mathfrak{Z}), 2\tau(\mathfrak{Z}) + 1\}$. If F is a general enough element of $(I_{\mathfrak{Z}})_j^\perp$, then F has a unique GAD of length s associated to \mathfrak{Z} , and no GAD's of length less than s .*

Proof. Let $t = \lfloor j/2 \rfloor$. By Theorem 3.3 the hypotheses on F and \mathfrak{Z} imply that $H_F = \text{Sym}(H_{\mathfrak{Z}}, j)$. Furthermore, the assumption on j implies that $(H_F)_t = (H_F)_{t+1} = s$, so $H_F \supset (s, s)$. It follows that any scheme \mathfrak{Z}' of degree at most s , such that $I_{\mathfrak{Z}'} \subset I_F$, satisfies $(H_{\mathfrak{Z}'})_t = (H_{\mathfrak{Z}'})_{t+1} = s$, hence $(I_{\mathfrak{Z}'})_t = (I_F)_t$ and $(I_{\mathfrak{Z}'})_{t+1} = (I_F)_{t+1}$. By Theorem 1.12 this equality implies that \mathfrak{Z}' is regular in degree $t+1$ (so $\sigma(\mathfrak{Z}') \leq t+1$), and is determined by F , so we must have $\mathfrak{Z} = \mathfrak{Z}'$. Uniqueness of the GAD now follows from Lemma 3.20. \square

Example 3.23. Let $R = k[x, y, z]$ and denote by Υ the degree 3 scheme $\Upsilon = \text{Proj}(R/(x, y^3))$ concentrated at the origin $p_0 = (0 : 0 : 1)$ of \mathbb{P}^2 ; and denote by $\mathfrak{Z} = \mathfrak{Z}_1 \cup \dots \cup \mathfrak{Z}_k$ the union of k distinct subschemes, where \mathfrak{Z}_i denotes a translation of Υ to a point $p_i = (a_{i0} : a_{i1} : 1) \in \mathbb{P}^2$ (by T_{p_i} as in Lemma 2.22). By Theorem 2.24, we have that $\mathfrak{Z}_i = \text{Proj}(R/(x - a_{i0}z, (y - a_{i1}z)^3))$, and since the inverse system $L(\Upsilon) \subset \Gamma = k_{DP}[X, Y, Z]$ satisfies $L(\Upsilon)_u = (I_\Upsilon)^\perp = R_u \circ (Y^{[2]} \cdot Z^u) = \langle Y^{[2]}Z^{[u-2]}, YZ^{[u-1]}, Z^{[u]} \rangle$, we have

$$\begin{aligned} L(\mathfrak{Z}_i)_u &= R \circ (Y^{[2]} \cdot (a_{i0}X + a_{i1}Y + Z)^{[u]}) \\ &= \langle Y^{[2]} \cdot (a_{i0}X + a_{i1}Y + Z)^{[u-2]}, Y \cdot (a_{i0}X + a_{i1}Y + Z)^{[u-1]}, (a_{i0}X + a_{i1}Y + Z)^{[u]} \rangle. \end{aligned}$$

Taking $k = 2$, letting $p_1 = p_0, p_2 = (1 : 1 : 1)$, we have

$$\mathcal{I}_{\mathfrak{Z}} = (x, y^3) \cap (x - z, (y - z)^3) = (x^2 - xz, 3xy^2 - y^3 - 3xyz + xz^2),$$

and $\Delta H_{\mathfrak{Z}} = (1, 2, 2, 1), H_{\mathfrak{Z}} = (1, 3, 5, 6, 6, \dots)$. By Lemma 3.20, a form $F \in (I_{\mathfrak{Z}})_j^\perp$ for $j \geq 3$ has a unique decomposition associated to \mathfrak{Z} , into $k = 2$ parts, each of length 3,

$$\begin{aligned} F &= F_1 + F_2 \text{ where } F_1 \in \langle Y^{[2]}Z^{[j-2]}, YZ^{[j-1]}, Z^{[j]} \rangle, \\ F_2 &\in \langle Y^{[2]} \cdot (X + Y + Z)^{[j-2]}, Y(X + Y + Z)^{[j-1]}, (X + Y + Z)^{[j]} \rangle. \end{aligned}$$

When $j = 3$, the form $F = 3Y^{[3]} + XY^{[2]} \in L_{\mathfrak{Z}}$, as it is evidently annihilated by $\mathcal{I}_{\mathfrak{Z}}$ acting as contraction. Thus F has a GAD with two summands, each of length 3,

$$F = Y^{[2]} \cdot (X + Y + Z) - Y^{[2]}Z. \quad (3.6)$$

By Theorem 3.22, since when $k = 2$, \mathfrak{Z} is Gorenstein with $\tau(\mathfrak{Z}) = 3$ and $\alpha(\mathfrak{Z}) = 2$, we have for $j \geq 7$ that a general $F \in (L_{\mathfrak{Z}})_j$ has a tight annihilating scheme \mathfrak{Z} , so a unique GAD of length 6. However, if $j \geq 6$, and F includes the terms $Y^{[2]}(X + Y + Z)^{[j-2]}$ and $Y^{[2]}Z^{[j-2]}$, then it is easily seen that F determines \mathfrak{Z} , as $I_{\mathfrak{Z}}$ is generated in degree 3, and $H_F = \text{Sym}(H_{\mathfrak{Z}}, j)$ by calculation.

Taking $k = 3$, using translates of Υ at the three points p_1, p_2 , and $p_3 = (2, 3, 1)$ we find $\Delta H_{\mathfrak{Z}} = (1, 2, 3, 3)$; taking $k = 4$ and points p_1, p_2, p_3 , and $p_4 = (7, 11, 1)$ we find $\Delta H_{\mathfrak{Z}} = (1, 2, 3, 4, 2)$. However, if we take instead $p'_4 = (2, 5, 1)$ we find $\Delta H_{\mathfrak{Z}'} = (1, 2, 3, 3, 2, 1)$. We might ask, for a generic choice of k points $\{p_i\}$, do we obtain a degree $3k$ scheme \mathfrak{Z} in *general position* - having the same Hilbert function as $3k$ generic smooth points? This is not the case for $k = 2$ here, but is true for $k = 3, 4$, and presumably for higher k .

Also, we may ask, what is the dimension of the family $\mathcal{F}(\Upsilon, k, \mathbf{P}^2)$ of all degree $3k$ punctual subschemes of \mathbb{P}^2 having the form $\mathfrak{Z} = \mathfrak{Z}_1 \cup \dots \cup \mathfrak{Z}_k$, with \mathfrak{Z}_i a translate of Υ ? In this direction, the tangent space to such families have been studied classically for power sum representations $F = \sum L_{p_i}^{[j]}$ (see [Ter2, Bro, AlH, I3], [IK, §2.1, 2.2], and for GAD's see [Eh, Tes], also [Cha2]).

Remark 3.24. In a sequel paper [ChoI2] we determine the global Hilbert functions $H_{\mathfrak{Z}}$ for compressed Gorenstein subschemes $\mathfrak{Z} \subset \mathbb{P}^n$. Let $r = n + 1$ and denote by $H_s(r)$ the sequence $H_s(r)_i = \min\{\dim_k R_i, s\}$. Then $H_s(r)$ is the global Hilbert function of a generic degree- s smooth scheme. We will show in the sequel that if \mathfrak{Z} is a general enough compressed local Gorenstein scheme of degree s , then $H_{\mathfrak{Z}} = H_s(r)$. Using Theorem 3.3, we will exhibit families $\mathbb{P}\text{GOR}(T)$

of graded Gorenstein Artin algebras of embedding dimension r and certain Hilbert functions $T = H(s, j, r) = \text{Sym}(H_s(r), j)$, $r \geq 5$, s large enough given r , that contain several irreducible components. Each component is fibred over a family of Gorenstein zero-dimensional schemes, with fibre an open in a projective space \mathbb{P}^{s-1} . One component is fibred over general enough smooth schemes $\mathfrak{Z} \subset \mathbb{P}^n$, $n = r - 1$ of degree s . The other component is fibred over a family of compressed Gorenstein subschemes. Here $T = H(s, j, r) = \text{Sym}(H_s(r), j)$ and $H(s, j, r)_i = \min\{r_i, r_{j-i}, s\}$ is the Hilbert function of GA algebras $R/\text{Ann}(F)$, $F = L_1^{[s]} + \cdots + L_s^{[j]} \in \Gamma$, determined by a dual generator F that is a sum of s general enough (divided) powers of linear forms. Some of these results were reported in [IK, §6.4].

Acknowledgements

We appreciated suggestions for the earlier version of Tony Geramita and the late Ruth Michler, of V. Kanev concerning the GAD viewpoint, discussions with Yuri Berest on projective closure and with Peter Schenzel concerning the connections with dualizing modules, and as well the comments of Bae-Eun Jung. The second author gratefully acknowledges the influence of many discussions begun some years ago with Jacques Emsalem concerning inverse systems and “points épais” [Em].

References

- [Ab] L. Abrams, *Two-dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory Ramifications **5** (1996), no. 5, 569 - 587.
- [AlH] J. Alexander and A. Hirschowitz, *Polynomial interpolation in several variables*, J. Algebraic Geometry **4** (1995), 201–222.
- [BGS] D. Bayer, D. Grayson, and M. Stillman, *Macaulay, A system for computation in algebraic geometry and commutative algebra*, 2000, <http://www.math.columbia.edu/~bayer/Macaulay/>.
- [BeRa] A. Bernardi and K. Ranestad, *The cactus rank of cubic forms*, Preprint, 2011, ArXiv arXiv:1110.2197.
- [BeI] D. Bernstein and A. Iarrobino, *A nonunimodal graded Gorenstein Artin algebra in codimension five*, Comm. Algebra **20** (1992), no. 8, 2323–2336.
- [Bo1] M. Boij, *Graded Gorenstein Artin algebras whose Hilbert functions have a large number of valleys*, Comm. Algebra **23** (1995), no. 1, 97–103.
- [Bo2] ———, *Gorenstein Artin algebras and points in projective space*, Bull. London Math. Soc. **31** (1999), no. 1, 11–16.
- [Bo3] ———, *Components of the space parametrizing graded Gorenstein Artin algebras with a given Hilbert function*, Pacific J. Math. **187** (1999), no. 1, 1–11.
- [BoL] M. Boij and D. Laksov, *Nonunimodality of graded Gorenstein Artin algebras*, Proc. Amer. Math. Soc. **120** (1994), 1083–1092.
- [Bra] J. Brachat, *Schémas de Hilbert, Décomposition de tenseurs*, Thèse, Univ. de Nice-Sophia-Antipolis, Juillet, 2011.
- [BrOt] M.C. Brambilla and G. Ottaviani, *On the Alexander-Hirschowitz theorem* J. Pure Appl. Algebra **212** (2008), no. 5, 1229–1251.

- [BS] M. Brodman and R. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge Studies in Advanced Mathematics 60, Cambridge University Press, Cambridge, 1998.
- [Bro] J. Bronowski, *The sums of powers as canonical expression*, Proc. Cambridge Philos. Soc. **29** (1933), 69-82.
- [BH] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics 39, Cambridge University Press, Cambridge, U.K., 1993; revised paperback edition, 1998.
- [BuB] W. Buczyńska and J. Buczyński, *Secant varieties to high degree Veronese reembeddings, catalecticant matrices, and smoothable Gorenstein schemes*, preprint, to appear J. Alg. Geometry. arXiv:1012.3563 (v.4 November 2011).
- [Cha1] K. Chandler, *Higher infinitesimal neighborhoods*, J. Algebra **205** (1998), 460-479.
- [Cha2] ———, *A brief proof of the maximal rank theorem for generic double points in projective space*, Trans. Amer. Math. Soc. **753** (2001), no. 5, 1907-1920.
- [CharP] M. Chardin and P. Philippon, *Regularité et interpolation*, J. Algebraic Geom. **8** (1999), 471-481.
- [ChoI1] Y. Cho and A. Iarrobino, *Hilbert functions and level algebras*, J. Algebra **241** (2001), no. 2, 745-758.
- [ChoI2] ——— and ———, *Gorenstein Artin algebras arising from punctual schemes in projective space*, preprint.
- [CR] C. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, J. Wiley, London, 1962.
- [Eh] R. Ehrenborg, *On apolarity and generic canonical forms*, J. Algebra **213** (1999), no. 1, 167-194.
- [EhR] R. Ehrenborg and G.-C. Rota, *Apolarity and canonical forms for homogeneous polynomials*, European J. of Combin. **14** (1993), 157-182.
- [Ei] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*. Graduate Texts in Math. 150, Springer-Verlag, Berlin and New York, 1994.
- [ElMo] M. Elkadi and B. Mourrain, *Introduction à la résolution des systèmes polynomiaux*, Mathématiques & Applications (Berlin) , 59. Springer, Berlin, 2007. vi+305 pp. ISBN: 978-3-540-71646-4; 3-540-71646-7.
- [Em] J. Emsalem, *Géométrie des Points Epais*, Bull. Soc. Math. France **106** (1978), 399-416.
- [EmI] J. Emsalem and A. Iarrobino, *Inverse system of a symbolic power I* , J. Algebra. **174** (1995), 1080-1090.
- [FL] R. Fröberg and D. Laksov, *Compressed algebras*, Conf. on Complete Intersections in Acireale, (S.Greco and R. Strano, eds.), Lecture Notes in Math. 1092, Springer-Verlag, Berlin and New York, 1984, 121-151.
- [Ge] A. V. Geramita, *Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals*, The Curves Seminar at Queen's, Vol X, (Kingston, 1995), Queen's Papers in Pure and Appl. Math., **102** (1996), 1-114.

- [GeM] A. V. Geramita and P. Maroscia, *The ideal of forms vanishing at a finite set of points in \mathbb{P}^n* , J. Algebra, **90** (1984), 528-555.
- [Ha] M. Haiman, *Conjectures on the quotient ring by diagonal invariants*, J. Algebraic Combin. **3** (1994), no. 1, 17-76.
- [Har] T. Harima, *Characterization of Hilbert functions of Gorenstein Artin algebras with the weak Stanley property*, Proc. Amer. Math. Soc. **123** (1995), 3631-3638.
- [I1] A. Iarrobino, *Compressed algebras: Artin algebras having given socle degrees and maximal length*, Trans. Amer. Math. Soc. **285** (1984), 337-378.
- [I2] ———, *Associated graded algebra of a Gorenstein Artin algebra*, Mem. Amer. Math. Soc. **107** (1994) no. 514.
- [I3] ———, *Inverse system of a symbolic power II: The Waring problem for forms*, J. Algebra **174** (1995), 1091-1110.
- [I4] ———, *Inverse system of a symbolic power III: thin algebras and fat points*, Compositio Math. **108** (1997), 319-356.
- [IK] A. Iarrobino and V. Kanev, *Power Sums, Gorenstein Algebras, and Determinantal Loci*, Lecture Notes in Mathematics 1721, Springer, Heidelberg, 1999.
- [IK1] A. Iarrobino and S. Kleiman, *The Gotzmann theorems and the Hilbert scheme*, Appendix C. in A. Iarrobino and V. Kanev, Power Sums, Gorenstein Algebras, and Determinantal Loci, Lecture Notes in Mathematics 1721, Springer, Heidelberg, 1999, 289-312.
- [K11] Jan O. Kleppe, *Maximal families of Gorenstein algebras*, Trans. Amer. Math. Soc. **358** (2006), no. 7, 3133–3167.
- [K12] ———, *Families of Artinian and one-dimensional algebras*, J. Algebra **311** (2007), no. 2, 665–701.
- [Kr1] M. Kreuzer, *On 0-dimensional complete intersections*, Math. Ann. **292** (1992), 43-58.
- [Kr2] ———, *Some applications of the canonical module of a zero-dimensional scheme*, Zero-dimensional Schemes, F. Orecchia and L. Chiantini, Walter de Gruyter, Berlin, 1994, 243-252.
- [La] D. Laksov, *Inverse systems, generic linear forms, and divided powers*, preprint, KTH, Stockholm, 2006, revised September 2011.
- [L-J] M. Lejeune-Jalabert, *Liaison et résidu*, in Algebraic geometry (La Rábida, 1981), Lecture Notes in Mathematics 961, Springer, Berlin-New York, 1982, 233-240.
- [Mac] F. H. S. Macaulay, *The Algebraic Theory of Modular Systems*, Cambridge Univ. Press, Cambridge, U. K., 1916; reprinted with a foreword by P. Roberts, Cambridge Univ. Press, London and New York, 1994.
- [Mar] P. Maroscia, *Some problems and results on finite sets of points in \mathbb{P}^n* , Open Problems in Algebraic Geometry, VIII, Proc. Conf. at Ravello, (C. Ciliberto, F. Ghione, and F. Orecchia, eds.) Lecture Notes in Mathematics 997, Springer-Verlag, Berlin and New York, 1983, 290-314.
- [MeSm] D. M. Meyer and L. Smith, *Poincaré duality algebras, Macaulay's dual systems, and Steenrod operations*, Cambridge Tracts in Mathematics, 167. Cambridge University Press, Cambridge, 2005. viii+193 pp.

- [Mi] R. Michler, *The dual of the torsion module of differentials*, Comm. Alg. **30** (2002), no. 12, 5639–5650.
- [Mig1] J. Migliore, *Introduction to Liaison Theory and Deficiency Modules*, Progress in Math. 165, Birkhäuser, Boston, 1998.
- [Mig2] ———, *Families of reduced zero-dimensional schemes*, Collect. Math. 57 (2006), no. 2, 173–192.
- [MigN] J. Migliore and U. Nagel, *Reduced arithmetically Gorenstein schemes and simplicial polytopes with maximal Betti numbers*, Adv. Math. 180 (2003), 1–63.
- [No] D. G. Northcott, *Injective envelopes and inverse polynomials.*, J. London Math. Soc. **8** (1974), no. 2, 290–296.
- [NR] D. G. Northcott and D. Rees, *Principal systems*, Quart. J. Math. Oxford **8** (1957), no. 2, 119–127.
- [Or] F. Orecchia, *One-dimensional local rings with reduced associated graded rings, and their Hilbert functions*, Manuscripta Math. **32** (1980), no. 3-4, 391–405.
- [RS1] K. Ranestad and F.-O. Schreyer, *The variety of polar simplices*, arXiv:1104.2728 (2011).
- [RS2] ——— and ———, *On the rank of a symmetric form*, arXiv:1104.3648 (2011).
- [Rez] B. Reznick, *Homogeneous polynomial solutions to constant coefficient PDE's*, Adv. Math. **117** (1996), no. 2, 179–192.
- [Schk] H. Schenck, *Elementary modifications and line configurations in \mathbb{P}^2* , Comment. Math. Helv. 78 (2003), no. 3, 447–462.
- [Schz] P. Schenzel, *Local cohomology and duality*, Six lectures on commutative algebra (Bellaterra, 1996), Progress in Math. 166, Birkhäuser, Basel, (1998), 241–292.
- [SmSt] L. Smith and R. E. Stong, *Projective bundle ideals and Poincaré duality algebras*. J. Pure Appl. Algebra 215 (2011), no. 4, 609–627.
- [Ter1] A. Terracini, *Sulle V_k per cui la varietà degli $S_h(h+1)$ -secanti ha dimensione minore dell'ordinario*, Rend. Circ. Mat. Palermo **31** (1911), 527–530.
- [Ter2] ———, *Sulla rappresentazione delle coppie di forme ternarie mediante somme di potenze di forme lineari*, Annali di Mat. Pura Appl. Serie III, **24** (1915), 1–10.
- [Tes] O. Teschke, *Moduli of rank two bundles with Chern classes $(0, 2)$ on quartic double solids*, Sitzungsberichte der Berliner Mathematischen Gesellschaft, Berliner Mathematischen Gesellschaft, Berlin, 2001, 181–200.
- [Wa] J. Watanabe, *The Dilworth number of Artin Gorenstein rings*, Adv. Math. **76** (1989), 194–199.
- [Za] F. Zanello, *The h -vector of a relatively compressed level algebra*, Comm. Algebra 35 (2007), no. 4, 1087–1091.
- [ZarS] O. Zariski and P. Samuel, *Commutative Algebra, Vol II.*, D. Van Nostrand Co, Princeton 1960.

e-mail addresses: youngcho@math.snu.ac.kr and a.iarrobino@neu.edu